

# Construction of infinite families of non-Schurian association schemes of order $2p^2$ , $p$ an odd prime, based on biaffine planes and Heisenberg groups: research report and beyond

ŠTEFAN GYÜRKI<sup>\*1</sup> AND MIKHAIL KLIN<sup>1,2</sup>

<sup>1</sup>Matej Bel University  
974 11 Banská Bystrica, Slovak Republic

<sup>2</sup>Department of Mathematics  
Ben-Gurion University of the Negev  
84105 Beer Sheva, Israel

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## Abstract

Let  $p$  be an odd prime. In this paper we provide a construction which gives four non-Schurian association schemes for every  $p \geq 5$  and two for  $p = 3$ . This construction is explained using incidences between points and lines of a biaffine plane and we also provide a pure algebraic model for it with the aid of finite Heisenberg groups. The obtained results are discussed in a more wide framework.

**Keywords.** Coherent configuration, association scheme, biaffine plane, computer algebra, Heisenberg group, non-Schurian scheme, plausible reasonings.

**MSC.** 05E30, 51E15.

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## 1 Introduction

This paper reports about research conducted on the edge between Algebraic Graph Theory (briefly AGT) and Computer Algebra.

Association schemes are one of the traditional areas of investigation in AGT. For a good decade, catalogues of small association schemes have been available from the web site [20]. It is known that all association schemes of order up to 14 are Schurian, that is, they are coming from a suitable transitive permutation group in the standard manner. First examples of non-Schurian association schemes exist on 15, 16 and 18 vertices. In particular, there are just two classes of non-Schurian association schemes of order 18.

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<sup>\*</sup>e-mail addresses: [gyurki@savbb.sk](mailto:gyurki@savbb.sk) (Š. Gyürki), [klin@cs.bgu.ac.il](mailto:klin@cs.bgu.ac.il) (M. Klin)

D. Pasechnik explained in evident form in [36] how the non-Schurian rank 3 antisymmetric association scheme on 15 points appears, and determined its full automorphism group.

The famous Shrikhande graph generates non-Schurian rank 3 association schemes on 16 points. In fact, there are many other non-Schurian schemes of order 16 (see [20]). Their clever computer free explanation might be of definite interest.

Surprisingly, such an explanation has not been given for the schemes on 18 points yet. We are here filling this gap, providing an interpretation in terms of finite geometries. Moreover, we are introducing a possible generalisation of these schemes which is leading to four infinite families of non-Schurian association schemes on  $2p^2$  points, where  $p > 3$  is a prime.

This paper originated from computer aided experiments, which were fulfilled by the author Š.Gy. under the guidance of M.K. The manner in which the computer search helped to reach the presented results is, in our eyes, of independent interest. The suggested structure of our paper was designed intentionally not only to reflect the obtained scientific results, but also to pursue additional pedagogical, expository and even philosophical goals.

Preliminaries and the used computer tools are briefly discussed in Sections 2 and 3, respectively. Our starting geometrical object is a biaffine coherent configuration  $\mathcal{M}$ , which appears via the intransitive permutation group  $H \cong \mathbb{Z}_p^2 \rtimes \mathbb{Z}_p$  of order  $p^3$  and degree  $2p^2$  having two orbits of length  $p^2$ . Object  $\mathcal{M}$  is considered in Sections 4–8, along with its four related color graphs  $\mathcal{M}_i$ ,  $1 \leq i \leq 4$ , which turn out to be association schemes. In particular, an outline of a proof that these association schemes are non-Schurian is given in Section 8.

The presentation in Sections 4–8 is of a definite geometric nature, based on a consideration of the classical biaffine plane of order  $p$  with  $p^2$  points and  $p^2$  lines.

In Section 9, all discussed structures are developed from the scratch with the aid of an independent second model of  $\mathcal{M}$ , which is of a more algebraic nature. Here our starting structure is the finite Heisenberg group of order  $p^3$ .

Sections 10 and 11 are devoted to the use of algebraic groups of coherent configurations. Here we suggest some innovative methodological elements, which might be of an independent interest for experts in AGT.

Sections 12 and 13 are aimed to extend the scope of our consideration, paying special attention to some extra features of our results, as well as to a number of well known classes of graphs which appear to be relevant to new association schemes discovered by the authors. Some of the considered graphs play a significant role in Extremal Graph Theory.

Section 14 serves as a research announcement of another portion of achieved results regarding new non-Schurian association schemes of constant rank 6 and 5, and some of their properties. The level of rigour here is different from the main body of the text. In particular, no attempt is made to provide full formulations and outlines of proofs.

Finally, the last Section 15 is written in a very specific style, quite typical of other publications co-authored by M.K. It is, in a sense, a mosaic of diverse topics, implicitly or explicitly related to the content of the paper, however not previously touched in evident form. Special attention is paid to the research paradigm “computer experiment, plausible reasonings, theoretical proof”, comparing views, experience, and tastes of the authors with modern trends in science, which are correlated with the exploitation of computer tools.

Appendices 1–3 contain some routine data about the investigated schemes. These data will play an essential role in portions of the theoretical proofs.

## 2 Preliminaries

Below we provide a brief outline of the most significant concepts that will be used throughout the text. We refer to [14] and [26] for a more detailed background.

By a *color graph*  $\Gamma$  we will mean an ordered pair  $(V, \mathcal{R})$ , where  $V$  is a set of vertices and  $\mathcal{R}$  a partition of  $V \times V$  into binary relations. The elements of  $\mathcal{R}$  will be called *colors*, and the number of colors will be the *rank* of  $\Gamma$ . In other words, a color graph is an edge-colored complete directed graph with loops, whose arcs are colored by the same color if and only if they belong to the same binary relation.

A *coherent configuration* is a color graph  $\mathcal{W} = (\Omega, \mathcal{R})$ ,  $\mathcal{R} = \{R_i \mid i \in I\}$ , such that the following axioms are satisfied:

- (i) The diagonal relation  $\Delta_\Omega = \{(x, x) \mid x \in \Omega\}$  is a union of relations  $\cup_{i \in I'} R_i$ , for a suitable subset  $I' \subseteq I$ .
- (ii) For each  $i \in I$  there exists  $i' \in I$  such that  $R_i^T = R_{i'}$ , where  $R_i^T = \{(y, x) \mid (x, y) \in R_i\}$  is the relation transposed to  $R_i$ .
- (iii) For any  $i, j, k \in I$ , the number  $c_{i,j}^k$  of elements  $z \in \Omega$  such that  $(x, z) \in R_i$  and  $(z, y) \in R_j$  is a constant depending only on  $i, j, k$ , and independent of the choice of  $(x, y) \in R_k$ .

The numbers  $c_{i,j}^k$  are called *intersection numbers*, or sometimes *structure constants* of  $\mathcal{W}$ .

Assume that  $|\Omega| = n$ , and let us put  $\Omega = \{1, 2, \dots, n\}$ . To each *basic graph*  $\Gamma_i = (\Omega, R_i)$  we associate its adjacency matrix  $A_i = A(\Gamma_i)$ . Then the set of *basic matrices*  $\{A_i \mid i \in I\}$  may be regarded as a basis of a matrix algebra  $\mathcal{H}$  which contains the identity matrix, the all-ones matrix  $J$ , and is closed under transposition and Schur-Hadamard multiplication of matrices. Such an algebra is called a *coherent algebra*, and we refer to the set  $\{A_i \mid i \in I\}$  as its *standard basis*.

The concepts of coherent configuration and coherent algebra were introduced by D. Higman (see e.g. [22]). Similar concepts were introduced independently by B.Ju. Weisfeiler and A.A. Leman, see [45] and also [28] for an historical discussion.

A significant source of coherent configurations appears as follows. Assume that  $(G, \Omega)$  is a permutation group acting on the set  $\Omega$ . For  $(\alpha, \beta) \in \Omega^2$  the set  $\{(\alpha, \beta)^g \mid g \in G\}$ , where  $(\alpha, \beta)^g = (\alpha^g, \beta^g)$ , is called a *2-orbit* of  $G$ , specifically the *2-orbit* of  $G$  corresponding to  $(\alpha, \beta)$ . (Note that when  $(G, \Omega)$  is a transitive permutation group, many authors prefer the term *orbital* for this set.)

Denoting by  $2\text{-Orb}(G, \Omega)$  the set of 2-orbits of a permutation group  $(G, \Omega)$ , it is easy to check that  $(\Omega, 2\text{-Orb}(\Omega))$  is a coherent configuration. Coherent configurations that arise in this manner are called *Schurian*, otherwise we call them *non-Schurian*.

An *association scheme*  $\mathcal{W} = (\Omega, \mathcal{R})$  (also called a *homogeneous coherent configuration*) is a coherent configuration in which the diagonal relation  $\Delta_\Omega$  belongs to  $\mathcal{R}$ . Thus, Schurian association schemes are coming from transitive permutation groups.

A coherent configuration  $\mathcal{W}$  is called *commutative* if for all  $i, j, k \in I$  we have  $c_{ij}^k = c_{ji}^k$ . We call  $\mathcal{W}$  *symmetric* if  $R_i = R_i^T$  for all  $i \in I$ . It is a well known fact that a symmetric coherent configuration is also commutative, but the converse is not true in general.

To each coherent configuration  $\mathcal{W}$  we may assign three groups:  $\text{Aut}(\mathcal{W})$ ,  $\text{CAut}(\mathcal{W})$  and  $\text{AAut}(\mathcal{W})$ . The (combinatorial) *group of automorphisms*  $\text{Aut}(\mathcal{W})$  consists of the permutations

$\phi : \Omega \rightarrow \Omega$  which preserve the relations, i.e.  $R_i^\phi = R_i$  for all  $R_i \in \mathcal{R}$ . The *color automorphisms* are permitted to permute the relations from  $\mathcal{R}$ , i.e. for  $\phi : \Omega \rightarrow \Omega$  we have  $\phi \in \text{CAut}(\mathcal{W})$  if and only if for all  $i \in I$  there exists  $j \in I$  such that  $R_i^\phi = R_j$ . An *algebraic automorphism* is a bijection  $\psi : \mathcal{R} \rightarrow \mathcal{R}$  that satisfies  $c_{ij}^k = c_{i\psi_j\psi}^{k\psi}$ . It is easy to verify that  $\text{Aut}(\mathcal{W})$  is a normal subgroup of  $\text{CAut}(\mathcal{W})$ , and that the quotient group  $\text{CAut}(\mathcal{W})/\text{Aut}(\mathcal{W})$  embeds naturally in  $\text{AAut}(\mathcal{W})$ .

For each group  $K$  of algebraic automorphisms of  $\mathcal{W} = (\Omega, \mathcal{R})$  one can define an *algebraic merging* of  $\mathcal{R}$  in the following way. Let  $\mathcal{R}/K$  denote the set of orbits of  $K$  on  $\mathcal{R}$ . For each  $O \in \mathcal{R}/K$  define  $O^+$  to be the union of all relations from  $O$ . Then the set of relations  $\{O^+ \mid O \in \mathcal{R}/K\}$  forms a coherent configuration on  $\Omega$ . We will call it an *algebraic merging* of  $\mathcal{R}$  with respect to  $K$ . Note that if  $K \leq \text{CAut}(\mathcal{W})/\text{Aut}(\mathcal{W})$  and  $\mathcal{W}$  is Schurian, then the resulting merging is Schurian as well. In contrast, a merging with respect to a subgroup of  $K$  not contained in  $\text{CAut}(\mathcal{W})/\text{Aut}(\mathcal{W})$  may lead to a non-Schurian coherent configuration.

It is clear from the definitions that an association scheme  $\mathcal{W}$  is Schurian if and only if its rank coincides with the rank of its group of automorphisms  $\text{Aut}(\mathcal{W})$ .

### 3 Computer tools

This project heavily depends on the use of computer tools. Moreover, in a sense it can serve as a pattern for the use of computers in AGT. Indeed, we made a lot of fast computations; investigated and organized the obtained results; analyzed all relevant data; made conjectures based on these data; checked these conjectures for higher values of parameters of the considered series; transformed these conjectures to a suitable analytical form; transformed the obtained results from numerical to symbolic mode; created pictures and tables; and so on.

Here we are mainly working with coherent configurations and association schemes, as well as with the permutation groups related to them. For this purpose, in 1990–92 a computer package was created in Moscow as a result of the activities of I.A. Faradžev and the author M.K. This package goes by the name **COCO**, and was introduced in [13]; see also [14] for deeper consideration of the used methodology and algorithms. **COCO** is still very helpful for performing initial computational experiments.

Nevertheless, nowadays the mainstream of our computer aided activities is based on the use of the free software **GAP** [16] (Groups, Algorithms and Programming), in particular its share package **GRAPE** [42] which works in conjunction with **nauty** [32].

In addition, we strongly benefitted from the kind permission granted by Sven Reichard to use his unpublished package **COCO IIR** [40] (still under development) which operates under the **GAP** platform. The foremost goal of **COCO IIR** is to extend the scope of algorithms from **COCO**, relying on many new developments and fresh ideas adopted from modern computer algebra.

Last but not least, we are pleased to acknowledge a package of programs for computing with association schemes, written by Hanaki and Miyamoto. It is open-source software that may be freely downloaded from the homepage of the authors [20]. This package also works under **GAP**, and contains some tools that have not yet been implemented in **COCO IIR**.

## 4 A biaffine coherent configuration from the biaffine plane

Let  $p$  be an odd prime, and let  $\mathbb{Z}_p$  be the cyclic group of order  $p$ . Throughout this text, the set of nonzero elements in  $\mathbb{Z}_p$  will be denoted by  $\mathbb{Z}_p^*$ . Take two copies  $\mathcal{P}$  and  $\mathcal{L}$  of  $\mathbb{Z}_p \times \mathbb{Z}_p$ . The first copy  $\mathcal{P}$  is none other than the point set of the classical (Desarguesian) affine plane of order  $p$ . Each element  $P \in \mathcal{P}$  may be identified uniquely with a pair of coordinates of the form  $P = [x, y]$ . Thus, we refer to the elements of  $\mathcal{P}$  as *points*. Let  $\mathcal{L}$  be the set of "non-vertical" lines in the affine plane, i.e.  $\ell \in \mathcal{L}$  if and only if the equation for  $\ell$  may be expressed as  $y = k \cdot x + q$  for some  $k, q \in \mathbb{Z}_p$ . Each line  $\ell$  is determined uniquely by a pair  $\ell = (k, q)$ . In order to distinguish points and lines we will use square brackets for points and parentheses for lines.

This geometry, in which one parallel class of the affine plane has been removed (in our case, the class of "vertical" lines) goes by the well established name *biaffine plane*. We denote it by  $\mathcal{B}_p$  (see Figure 1 depicting  $\mathcal{B}_3$ ). For more details about biaffine planes and their relation to other combinatorial structures, we recommend [47]; see also additional comments at the end of this paper.

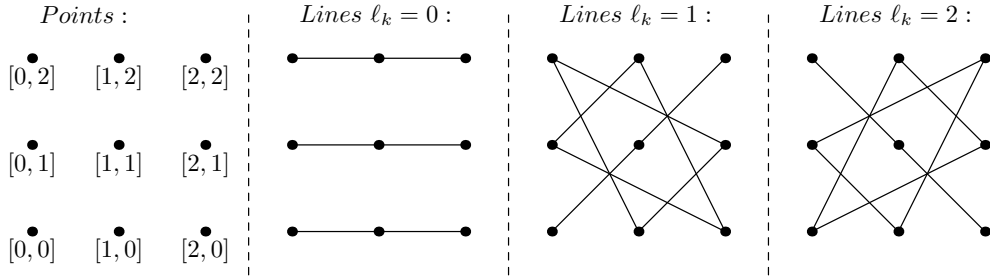


Figure 1: The objects of the biaffine plane  $\mathcal{B}_3$ .

Given a point  $P = [x, y]$  and a line  $\ell = (k, q)$  of the biaffine plane, we define a *quasidistance*  $d : (\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P}) \rightarrow \mathbb{Z}_p$  by the formulas:  $d(P, \ell) = k \cdot x + q - y$  and  $d(\ell, P) = y - q - k \cdot x$ . Note that  $d$  does not define a metric (indeed, it is not symmetric and non-negativity does not make sense in  $\mathbb{Z}_p$ ). Thus, it is just a vague analogue.

Recall that a natural source of coherent configurations is coming from permutation groups. If we take any permutation group  $(G, \Omega)$ , then  $\mathcal{W} = (\Omega, 2\text{-Orb}(G))$  is its associated (Schurian) coherent configuration. In particular, if  $(G, \Omega)$  is transitive, then  $\mathcal{W}$  is an association scheme.

Let us now consider an action of the group  $H = (\mathbb{Z}_p)^2 \rtimes \mathbb{Z}_p$  on the set  $\Omega = \mathcal{P} \cup \mathcal{L}$ , most conveniently described in terms of generators. To each pair  $(a, b) \in \mathbb{Z}_p^2$  we associate a *translation*  $t_{ab}$  acting naturally on  $\mathcal{P}$  as  $[x, y] \mapsto [x + a, y + b]$ , while the induced action on  $\mathcal{L}$  is  $(k, q) \mapsto (k, b + q - ak)$ . Of course, the set of all translations forms an Abelian group of order  $p^2$  under composition of permutations (denoted  $\circ$ ) and is isomorphic to  $\mathbb{Z}_p^2$ . Further, let  $\varphi : \mathcal{P} \rightarrow \mathcal{P}$  be defined by  $\varphi : [x, y] \mapsto [x, y - x]$ . The corresponding permutation on lines is  $(k, q) \mapsto (k - 1, q)$ . Clearly,  $\varphi$  is a permutation of order  $p$  and it is immediate that  $\varphi \circ t_{a,b-a} = t_{a,b} \circ \varphi$ . Our group  $H$  above is generated by all translations together with the permutation  $\varphi$ . Note that all elements of  $H$  may be expressed in the form  $\varphi^u \circ t_{a,b}$ , where  $a, b, u$  are suitable elements of  $\mathbb{Z}_p$ . Moreover, to distinct triples of  $a, b, u$  correspond distinct elements of  $H$ . In other words,  $H = \langle t_{a,b}, \varphi \rangle$  and  $|H| = p^3$ . Now set  $h_{a,b,u} := \varphi^u \circ t_{a,b}$ . Then multiplication in  $H$  is given by

$$h_{a,b,u} \circ h_{c,d,v} = h_{a+c,b+d-av,u+v}.$$

Observe that the action of  $H$  is intransitive on  $\Omega$  with two orbits  $\mathcal{P}$  and  $\mathcal{L}$ .

In what follows, the introduced permutation group  $(H, \Omega)$  will be called the *Heisenberg group modulo  $p$* ; see Section 9 for more details.

**Proposition 1** *Let  $p$  be an odd prime. Then*

- (1) *the rank of  $(H, \Omega)$  is  $6p - 2$ ,*
- (2) *the 2-orbits of  $(H, \Omega)$  may be divided into six different types of classes  $A_i, B_i, C_i, D_i, E_i$  and  $F_i$ , which are characterized by suitable relations between coordinates of objects in  $\Omega$ .*

**Proof.** Let  $P_1 = [x_1, y_1]$ ,  $P_2 = [x_2, y_2] \in \mathcal{P}$  and  $\ell_1 = (k_1, q_1)$ ,  $\ell_2 = (k_2, q_2) \in \mathcal{L}$ . Then the types of classes are the following:

- $(P_1, P_2) \in A_i \iff x_1 = x_2$  and  $y_2 - y_1 = i$ , where  $i \in \mathbb{Z}_p$  (i.e., those pairs of points whose first coordinates are equal and second coordinates differ by  $i$  in the given order; note that  $A_0$  is the diagonal relation on  $\mathcal{P}$ ),
- $(P_1, P_2) \in B_i \iff x_2 - x_1 = i$ , where  $i \in \mathbb{Z}_p^*$  (i.e., those pairs of points whose first coordinates differ by  $i$  in the given order),
- $(\ell_1, \ell_2) \in C_i \iff k_1 = k_2$  and  $q_2 - q_1 = i$ , where  $i \in \mathbb{Z}_p$  (i.e., those pairs of lines whose first coordinates are equal and second coordinates differ by  $i$  in the given order; note that  $C_0$  is the diagonal relation on  $\mathcal{L}$ ),
- $(\ell_1, \ell_2) \in D_i \iff k_2 - k_1 = i$ , where  $i \in \mathbb{Z}_p^*$  (i.e., those pairs of lines whose first coordinates differ by  $i$  in the given order),
- $(P_1, \ell_1) \in E_i \iff k_1 \cdot x_1 + q_1 - y_1 = i$ , where  $i \in \mathbb{Z}_p$  (i.e., those point-line pairs whose quasidistance  $d(P_1, \ell_1)$  is  $i$ ),
- $(\ell_1, P_1) \in F_i \iff y_1 - k_1 x_1 - q_1 = i$ , where  $i \in \mathbb{Z}_p$  (i.e., those point-line pairs whose quasidistance  $d(\ell_1, P_1)$  is  $i$ ).

To complete the proof, one must verify two things: that the  $6p - 2$  relations introduced above indeed form a partition of the set  $\Omega^2$ , and that each such relation is in fact a 2-orbit of  $(H, \Omega)$ .  $\square$

**Remark 1.** One can easily check from its definition that the permutation  $\varphi$  has exactly  $p$  fixed points in its action on  $\mathcal{P}$ , as well as  $p$  fixed points in its action on  $\mathcal{L}$ . Relying on the bijection between 2-orbits of a transitive permutation group and orbits (1-orbits) of the stabilizer of an arbitrary point (see e.g. [14]), the reader can easily deduce that there exists exactly  $p + (p - 1)$  2-orbits of the transitive action  $(H, \mathcal{P})$ , and similarly for the action  $(H, \mathcal{L})$ . Thus, we obtain in this manner  $4p - 2$  2-orbits of  $(H, \Omega)$ . Observing that there are  $p$  2-orbits of type  $E_i$  and  $p$  of type  $F_i$ , we arrive at the desired amount of  $6p - 2$ .

**Remark 2.** Reflexive 2-orbits  $A_0$  and  $C_0$  are obviously symmetric, however all remaining 2-orbits are antisymmetric. Namely, we obtain that  $A_i^T = A_{p-i}$ ,  $B_i^T = B_{p-i}$ ,  $C_i^T = C_{p-i}$ ,  $D_i^T = D_{p-i}$ ,  $E_i^T = F_{p-i}$  and  $F_i^T = E_{p-i}$ . Here and below, operations on subscripts are performed as in  $\mathbb{Z}_p$ .

This description provides a quite nice geometrical interpretation of the relations. In the sections to follow, we will define certain families of color graphs as suitable mergings of these relations. We will prove that these color graphs are in fact association schemes, and we will investigate them further.

Note that at the end of the paper we will once again justify a portion of our obtained results, this time relying on a more purely algebraic approach.

Denote by  $\mathcal{M}$  the coherent configuration corresponding to the group  $H$  in our construction above. For obvious reasons, we shall refer to  $\mathcal{M}$  as a *biaffine coherent configuration*.

## 5 Intersection numbers of the biaffine coherent configuration $\mathcal{M}$

The biaffine coherent configuration  $\mathcal{M}$  was constructed with the aid of a group, that is, it is Schurian. Presently, we are interested in the intersection numbers of  $\mathcal{M}$ . In this section we display these numbers with the aid of tables. In each individual table the superscript is fixed, the symbol in the row indicates the first subscript, and the symbol in the column indicates the second subscript.

We are using the *Kronecker's symbol*  $\delta_{i,j}$  in order to shorten computations and formulas.

**Proposition 2** *The tensor of structure constants of the biaffine coherent configuration  $\mathcal{M}$  is given as follows:*

$c_{r_i, c_j}^{A_k}$	$A_j$	$B_j$	$F_j$
$A_i$	$\delta_{i+j,k}$	0	0
$B_i$	0	$p \cdot \delta_{i+j,0}$	0
$E_i$	0	0	$p \cdot \delta_{i+j,k}$

$c_{r_i, c_j}^{C_k}$	$C_j$	$D_j$	$E_j$
$C_i$	$\delta_{i+j,k}$	0	0
$D_i$	0	$p \cdot \delta_{i+j,0}$	0
$F_i$	0	0	$p \cdot \delta_{i+j,k}$

$c_{r_i, c_j}^{B_k}$	$A_j$	$B_j$	$F_j$
$A_i$	0	$\delta_{j,k}$	0
$B_i$	$\delta_{i,k}$	$p \cdot \delta_{i+j,k}$	0
$E_i$	0	0	1

$c_{r_i, c_j}^{D_k}$	$C_j$	$D_j$	$E_j$
$C_i$	0	$\delta_{j,k}$	0
$D_i$	$\delta_{i,k}$	$p \cdot \delta_{i+j,k}$	0
$F_i$	0	0	1

$c_{r_i, c_j}^{E_k}$	$C_j$	$D_j$	$E_j$
$A_i$	0	0	$\delta_{i+j,k}$
$B_i$	0	0	1
$E_i$	$p \cdot \delta_{i+j,k}$	1	0

$c_{r_i, c_j}^{F_k}$	$A_j$	$B_j$	$F_j$
$C_i$	0	0	$\delta_{i+j,k}$
$D_i$	0	0	1
$F_i$	$p \cdot \delta_{i+j,k}$	1	0

where the indices  $i, j, k$  go through all feasible values. All structure constants not displayed here are zero.

**Example 1.** In the biaffine coherent configuration of order 50 (i.e.  $p = 5$ ) we have

$$\begin{aligned}
c_{A_2, A_4}^{A_1} &= \delta_{2+4,1} = \delta_{1,1} = 1, \\
c_{B_2, B_3}^{A_4} &= 5 \cdot \delta_{2+3,0} = 5 \cdot \delta_{0,0} = 5, \\
c_{C_2, A_4}^{A_3} &= 0, \\
c_{D_2, F_0}^{F_1} &= 1.
\end{aligned}$$

**Proof (outline).**

First observe that for all  $i \in \mathbb{Z}_p$  and  $j \in \mathbb{Z}_p^*$ , we have  $A_i, B_j \subseteq \mathcal{P} \times \mathcal{P}$ ,  $C_i, D_j \subseteq \mathcal{L} \times \mathcal{L}$ ,  $E_i \subseteq \mathcal{P} \times \mathcal{L}$  and  $F_i \subseteq \mathcal{L} \times \mathcal{P}$ . Thus all structure constants of the form  $c_{Y_i, Z_j}^{X_k}$  are zero provided  $X, Y, Z$  satisfy any of the following:

- $X \in \{A, B, E\}$  and  $Y \in \{C, D, F\}$ , or  $Y \in \{A, B, E\}$  and  $X \in \{C, D, F\}$ ,
- $X \in \{A, B, F\}$  and  $Z \in \{C, D, E\}$ , or  $Z \in \{A, B, F\}$  and  $X \in \{C, D, E\}$ ,
- $Y \in \{A, B, F\}$  and  $Z \in \{C, D, F\}$ , or  $Y \in \{C, D, E\}$  and  $Z \in \{A, B, E\}$ .

The reason is simply that in these cases the composition of relations  $Y_i$  and  $Z_j$  is either impossible, or giving a relation disjoint to  $X_k$ . This observation is crucial in order to understand that those structure constants which in principle may be nonzero are covered just by the six kinds of tables presented above.

Simple algebraic manipulations of coordinates lead us to the following:

$$c_{A_i, A_j}^{A_k} = c_{C_i, C_j}^{C_k} = \delta_{i+j, k}, \quad c_{B_i, B_j}^{A_k} = c_{D_i, D_j}^{C_k} = p \cdot \delta_{i+j, 0}, \quad c_{A_i, B_j}^{B_k} = c_{C_i, D_j}^{D_k} = \delta_{j, k},$$

$$c_{B_i, A_j}^{B_k} = c_{D_i, C_j}^{D_k} = \delta_{i, k}, \quad \text{and} \quad c_{B_i, B_j}^{B_k} = c_{D_i, D_j}^{D_k} = p \cdot \delta_{i+j, k}.$$

Computation of the remaining structure constants is not as straightforward and requires a bit more sophistication. Yet, these may be determined by counting with coordinates. For example, to show that  $c_{E_i, F_j}^{A_k} = p \cdot \delta_{i+j, k}$  we first consider a pair  $(P_1, P_2) \in A_k$ . Putting  $P_1 = [x_1, y_1]$ , this means that  $P_2$  is uniquely determined by the coordinates  $P_2 = [x_1, y_1 + k]$ . We now seek the number of lines  $\ell = (m, q)$  for which  $(P_1, \ell) \in E_i$  and  $(\ell, P_2) \in F_j$ . This yields two equations that must be satisfied by the coordinates of  $\ell$ :

$$m \cdot x_1 + q - y_1 = i$$

$$(y_1 + k) - m \cdot x_1 - q = j.$$

The sum of these two equations tells us that there are no solutions when  $i + j \neq k$  in  $\mathbb{Z}_p$ . To the contrary, if  $i + j = k$  in  $\mathbb{Z}_p$  then all lines that satisfy the first equation are automatically solutions to the entire system. In this manner we obtain precisely  $p$  solutions, one for each fixed choice of  $m \in \mathbb{Z}_p$ .

In a similar fashion one can derive all remaining structure constants of  $\mathcal{M}$ . □

## 6 Construction of four families of color graphs

Now we are ready to describe our four families of color graphs. To emphasize the connection between these color graphs and the initial biaffine coherent configuration  $\mathcal{M}$ , we define basic relations in terms of  $A_i, B_i, C_i, D_i, E_i$  and  $F_i$ . Later we will show that these color graphs are corresponding to association schemes.

Let us consider the following subsets of  $\Omega \times \Omega$ :

- $R_0 = A_0 \cup C_0$ ,



- $S_i = A_i \cup C_i$ , where  $i \in \mathbb{Z}_p^*$ ,
- $T_i = B_i \cup D_i$ , where  $i \in \mathbb{Z}_p^*$ ,
- $U_i = E_i \cup F_i$ , where  $i \in \mathbb{Z}_p$ . Note that the relation  $U_0$  coincides with the set of flags in the biaffine plane  $\mathcal{B}_p$ .

Further, let  $S_i^* = S_i \cup S_{p-i}$ ,  $T_i^* = T_i \cup T_{p-i}$  and  $U_i^* = U_i \cup U_{p-i}$  be the respective symmetrizations of the relations  $S_i$ ,  $T_i$  and  $U_i$ , canonically denoted for each  $i \in \{1, 2, \dots, (p-1)/2\}$ . Finally, let  $S = S_1 \cup S_2 \cup \dots \cup S_{p-1}$  and  $U = U_1 \cup U_2 \cup \dots \cup U_{p-1}$ . Observe that  $U$  is the set of antiflags in  $\mathcal{B}_p$ .

It is straightforward to check that  $\{R_0, S_1, \dots, S_{p-1}, T_1, \dots, T_{p-1}, U_0, U_1, \dots, U_{p-1}\}$  forms a partition of  $\Omega \times \Omega$ .

It remains to define the requested color graphs  $\mathcal{M}_i$ ,  $1 \leq i \leq 4$ , each with vertex set  $\Omega$ .

**Color graph 1.** Denote by  $\mathcal{M}_1$  the color graph with colors given by the sets  $R_0, S_1, \dots, S_{p-1}, T_1, \dots, T_{p-1}, U_0, U_1, \dots, U_{p-1}$ .

**Color graph 2.** Denote by  $\mathcal{M}_2$  the color graph with colors given by  $R_0, S_1^*, S_2^*, \dots, S_{(p-1)/2}^*, T_1, T_2, \dots, T_{p-1}, U_0, U_1^*, U_2^*, \dots, U_{(p-1)/2}^*$ .

**Color graph 3.** Denote by  $\mathcal{M}_3$  the color graph with colors given by  $R_0, S, T_1, T_2, \dots, T_{p-1}, U_0, U$ .

**Color graph 4.** Finally, denote by  $\mathcal{M}_4$  the color graph with colors given by  $R_0, S, T_1^*, T_2^*, \dots, T_{(p-1)/2}^*, U_0, U$ .

Note that for  $p = 3$  the color graphs  $\mathcal{M}_2$  and  $\mathcal{M}_3$  coincide.

These four color graphs will play an important role in this paper.

## 7 Intersection numbers

We wish now to show that our color graphs  $\mathcal{M}_1$ – $\mathcal{M}_4$  defined in the previous section are association schemes. Observe that it is sufficient to show the existence of intersection numbers (structure constants) because all other axioms of an association scheme are trivially satisfied.

For convenience, we shall invoke the following notational simplification for the indices of intersection numbers. We shall write  $si, ti, ui$  in place of  $S_i, T_i, U_i$ , respectively. For example,  $c_{si,tj}^{uk}$  indicates the number of elements  $z \in \Omega$  such that  $(x, z) \in S_i$  and  $(z, y) \in T_j$  for any  $(x, y) \in U_k$ . A subscripted or superscripted zero shall always indicate the relation  $R_0$ , while any index  $i$  not accompanied by a specified symbol will indicate any feasible relation. For the sake of brevity, we shall only indicate those intersection numbers in which starred relations (such as  $U_k^*$ ) occur in the superscript. In each case we are providing only one argumentation (usually for points), because the dual consideration (for lines) is similar. We make frequent use of *Kronecker's symbol*  $\delta_{i,j}$  in order to shorten computations and formulas.

To make enumeration easier the following observations are helpful:

**Observation 1.** For all  $1 \leq i \leq p-1$  and  $1 \leq j \leq (p-1)/2$  we have  $R_0, S, S_i, T_i, S_j^*, T_j^* \subseteq (\mathcal{P} \times \mathcal{P}) \cup (\mathcal{L} \times \mathcal{L})$ , and  $U, U_0, U_i, U_j^* \subseteq (\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P})$ . Thus, the intersection numbers of type  $c_{si,sj}^{uk}, c_{si,tj}^{uk}, c_{ti,sj}^{uk}, c_{ti,tj}^{uk}, c_{ui,uj}^{sk}, c_{si,uj}^{sk}, c_{si,uj}^{tk}, c_{ti,uj}^{sk}, c_{ti,uj}^{tk}, c_{ui,sj}^{sk}, c_{ui,sj}^{tk}, c_{ui,tj}^{sk}, c_{ui,tj}^{tk}$  are zero for all choices of  $i, j, k$ .

**Remark 3.** Let us choose the symbol  $*$  to indicate *composition* of relations. Recall that in a coherent algebra this operation corresponds to a product of corresponding adjacency matrices. In a coherent configuration the result of composition is usually a *multirelation*, that is, a set of relations together with their (non-negative) integer multiplicities. Thus we have adopted  $*$  in order to avoid misunderstandings, since the binary operator  $\circ$  is reserved for the Schur-Hadamard product in the theory of coherent configurations.

**Observation 2.** For the compositions of relations  $S_i, S_j, T_i, T_j$  we have  $S_i * T_j = T_j$ ,  $T_i * S_j = T_i$  and if  $i + j \neq 0$ , then  $S_i * S_j = S_{i+j}$  and  $T_i * T_j = T_{i+j}$ . As a consequence we obtain the following:  $c_{si,tj}^{sk} = c_{tj,si}^{sk} = c_{si,sj}^{tk} = 0$ , and for  $i + j \neq 0$ :  $c_{ti,tj}^{sk} = 0$ ,  $c_{si,sj}^{sk} = \delta_{i+j,k}$ .

**Observation 3.** For each color  $X$  we have  $R_0 * X = X * R_0 = X$ , and for any  $Y \neq X^T$  we obtain  $X * Y \neq R_0 \neq Y * X$ . Thus for  $i \neq j$ :  $c_{0,i}^j = c_{i,0}^j = 0$ , and for  $i \neq j'$ :  $c_{i,j}^0 = 0$ .

**Observation 4.** Let  $P_1, P_2 \in \mathcal{P}$  be two collinear points in  $\mathcal{B}_p$ , and let  $L_1 = \{\ell \in \mathcal{L} \mid P_1 \in \ell\}$ . Then for all  $\ell_i, \ell_j \in L_1$  we have  $d(P_2, \ell_i) = d(P_2, \ell_j)$  if and only if  $i = j$ , where  $d(P, \ell)$  is the previously defined quasidistance.

All intersection numbers are displayed in Appendix 1. These values were derived by geometrical arguments, usually by considering points and lines at a given quasidistance from two objects. For example, consider the color graph  $\mathcal{M}_1$ , and let  $P = [x, y]$  and  $i \in \mathbb{Z}_p$  be fixed. Then there is a unique point with coordinates  $[x, y + i]$ , precisely  $p$  points with first coordinate  $x + i$ , and precisely  $p$  lines at quasidistance  $i$  from  $P$ . These observations lead directly to the intersection numbers  $c_{si,sj}^0 = \delta_{i,j'}$ ,  $c_{ti,tj}^0 = p \cdot \delta_{i,j'}$  and  $c_{ui,uj}^0 = p \cdot \delta_{i,j'}$ , respectively.

**Theorem 3** *The following holds:*

- (a)  $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4$  are association schemes.
- (b) The combinatorial groups of automorphisms of  $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4$  contain a subgroup isomorphic to  $H$ .
- (c)  $\mathcal{M}_2$  is a merging of  $\mathcal{M}_1$ ,  $\mathcal{M}_3$  is a merging of  $\mathcal{M}_2$ , and  $\mathcal{M}_4$  is a merging of  $\mathcal{M}_3$ .
- (d)

$$\begin{aligned} \text{rk}(\mathcal{M}_1) &= 3p - 1, \\ \text{rk}(\mathcal{M}_2) &= 2p, \\ \text{rk}(\mathcal{M}_3) &= p + 3, \\ \text{rk}(\mathcal{M}_4) &= (p + 7)/2. \end{aligned}$$

**Proof.** Parts (a) and (b) have already been proven. Proofs of (c) and (d) follow easily from the definition of  $\mathcal{M}_i$ ,  $1 \leq i \leq 4$ . □

## 8 Automorphism groups of the association schemes

Recall that to each association scheme  $\mathcal{M}$  we may assign three groups:  $\text{Aut}(\mathcal{M})$ ,  $\text{CAut}(\mathcal{M})$  and  $\text{AAut}(\mathcal{M})$ . In this section we will focus on the combinatorial group of automorphisms  $\text{Aut}(\mathcal{M})$ .

Recall that this group consists of all permutations  $\phi : \Omega \rightarrow \Omega$  that preserve relations, i.e.  $R_i^\phi = R_i$  for all  $R_i \in \mathcal{R}$ .

**Theorem 4** *Let  $\text{Aut}(\mathcal{M}_1)$ ,  $\text{Aut}(\mathcal{M}_2)$ ,  $\text{Aut}(\mathcal{M}_3)$  and  $\text{Aut}(\mathcal{M}_4)$  be the combinatorial groups of automorphisms of  $\mathcal{M}_1$ ,  $\mathcal{M}_2$ ,  $\mathcal{M}_3$  and  $\mathcal{M}_4$ , respectively. Then the following hold:*

- (a)  $\text{Aut}(\mathcal{M}_1) \leq \text{Aut}(\mathcal{M}_2) = \text{Aut}(\mathcal{M}_3) \leq \text{Aut}(\mathcal{M}_4)$ ,
- (b)  $|\text{Aut}(\mathcal{M}_1)| = p^3$ ,
- (c)  $|\text{Aut}(\mathcal{M}_2)| = 2p^3$ ,
- (d)  $|\text{Aut}(\mathcal{M}_3)| = 2p^3$ ,
- (e)  $|\text{Aut}(\mathcal{M}_4)| = 8p^3$ .

**Proof.** It is clear that the previously defined permutations  $t_{ab}$  and  $\varphi$  are elements of each automorphism group  $\text{Aut}(\mathcal{M}_i)$ , hence  $H$  is a subgroup of  $\text{Aut}(\mathcal{M}_i)$  for each  $1 \leq i \leq 4$ .

- (a) The chain of inequalities  $\text{Aut}(\mathcal{M}_1) \leq \text{Aut}(\mathcal{M}_2) \leq \text{Aut}(\mathcal{M}_3) \leq \text{Aut}(\mathcal{M}_4)$  follows directly from Theorem 3, simply by applying Galois correspondence to the lattice of coherent configurations and that of their corresponding automorphism groups. The equality  $\text{Aut}(\mathcal{M}_2) = \text{Aut}(\mathcal{M}_3)$  will follow by inspection of the group orders, to be accomplished in parts (c) and (d) below.

Below we provide separate proofs for each of the claims (b) through (e). We use the same methodology throughout. In each proof,  $G$  will denote the group  $\text{Aut}(\mathcal{M}_i)$ , while  $G_{[0,0],(0,0)}$  will denote the stabilizer in  $G$  of both the point  $[0,0]$  and the line  $(0,0)$ . The final result is obtained via manipulation of suitable elements of  $G$  and application of the classical orbit-stabilizer lemma. Out of necessity, we will introduce certain suitable permutations acting on  $\mathcal{P} \cup \mathcal{L}$  that we have not already encountered.

- (b) We apply the orbit-stabilizer lemma to prove that  $|\text{Aut}(\mathcal{M}_1)| \leq p^3$ . As we already know that  $H$  is a subgroup of  $\text{Aut}(\mathcal{M}_1)$ , the result will follow. (In fact, this will further show that  $\text{Aut}(\mathcal{M}_1) \cong H \cong \mathbb{Z}_p^2 \rtimes \mathbb{Z}_p$ .)

Denote  $G := \text{Aut}(\mathcal{M}_1)$  for brevity. First we claim that there is no automorphism that sends a point to a line. By way of contradiction, suppose  $\alpha \in G$  sends the point  $P_1$  to the line  $r = (k, q)$ ,  $k, q \in \mathbb{Z}_p$ . Without loss of generality we may assume  $P_1 = [0, 0]$ , because  $G$  acts transitively on  $\mathcal{P}$ . In such case, we must have  $\mathcal{P}^\alpha = \mathcal{L}$  and  $\mathcal{L}^\alpha = \mathcal{P}$ , because of the relations  $S_i$ . Consider now the line  $l = (0, 0)$  and its image  $l_1^\alpha = (u, v)$ ,  $u, v \in \mathbb{Z}_p$ . Clearly  $(P_1, l_1) \in U_0$ , whence  $v = k \cdot u + q$ . Now consider the point  $[1, 0]$ . Since  $([0, 0], [1, 0]) \in T_1$  and  $([1, 0], (0, 0)) \in U_0$ , it follows that  $[1, 0]^\alpha = (k + 1, q - u)$ . Similarly,  $(1, 0)^\alpha = [u + 1, q + k(u + 1)]$ . However  $([1, 0], (1, 0)) \in U_1$ , and therefore  $([1, 0], (1, 0))^\alpha \in U_1$ . But  $([1, 0], (1, 0))^\alpha = ((k + 1, q - u), [u + 1, q + k(u + 1)]) \in U_{-1}$ , since  $q + k(u + 1) - (k + 1)(u + 1) - q + u = -1$ , a contradiction for any odd prime  $p$ . This proves that  $\mathcal{P}^G = \mathcal{P}$  and  $\mathcal{L}^G = \mathcal{L}$ , as claimed.

Since  $G$  is transitive on the points,  $|[0, 0]^G| = |\mathcal{P}| = p^2$ . Let  $G_{[0]} := G_{[0,0]}$  be the stabilizer in  $G$  of the point  $[0, 0]$ . The points  $[0, 1], [0, 2], \dots, [0, p - 1]$  are fixed by  $G_{[0]}$ , because they

form unique pairs together with  $[0, 0]$  in the relations  $S_1, S_2, \dots, S_{p-1}$ , respectively. As the line  $(0, 0)$  contains the point  $[0, 0]$ , there are at most  $p$  distinct images of  $(0, 0)$  under the action of  $G_{[0]}$ . However, it is easy to check that  $(0, 0)^{\varphi^i} = (-i, 0)$  for  $0 \leq i \leq p-1$ , which proves that  $|(0, 0)^{G_{[0]}}| = p$ .

Now let  $G_0 := G_{[0,0],(0,0)}$  be the stabilizer in  $G$  of both  $(0, 0)$  and  $[0, 0]$ . Then  $G_0$  fixes all lines  $(0, i)$  parallel to  $(0, 0)$ , because  $(0, i)$  forms a unique pair with  $(0, 0)$  in  $S_i$ . If we now consider an arbitrary point  $[x, y]$  with  $x \neq 0$ , then its image under  $G_0$  must be contained in the line  $(0, y)$ . Moreover,  $([0, y], [x, y]) \in T_x$  and for any  $\pi \in G_0$  it follows that  $([0, y], [x, y])^\pi = ([0, y], [t, y]) \in T_x$  for some  $t \in \mathbb{Z}_p$ . This establishes that  $t = x$ , and hence the point  $[x, y]$  is fixed under  $G_0$ . Thus  $G_0$  fixes all points and therefore all lines as well. By the orbit-stabilizer lemma, this yields  $|G| = p^2 \cdot p \cdot 1 = p^3$  and so  $G \cong H$  as desired.

- (c) By routine inspection, one can check that the mapping  $\pi$  defined by  $[x, y] \mapsto (x, -y - 2x)$ ,  $(x, y) \mapsto [x + 2, -y]$  is an automorphism of  $\mathcal{M}_2$ . From this it follows that  $G := \text{Aut}(\mathcal{M}_2)$  is transitive on  $V = \mathcal{P} \cup \mathcal{L}$ , whence  $|[0, 0]^G| = 2p^2$ . As in the case above, we may again show that the line  $(0, 0)$  has  $p$  distinct images under the action of the stabilizer  $G_{[0]}$  of  $[0, 0]$ . Let us again consider the stabilizer  $G_0$  of point  $[0, 0]$  and line  $(0, 0)$ . For any point  $[x, y] \in \mathcal{P}$  we have that  $[x, y]^{G_0} \subseteq \{[x, y], [x, -y]\}$  because of the relations  $S_i^*$  and  $U_i^*$ . If there exists some point  $P_1 = [x, y]$  ( $x \neq 0$  and  $y \neq 0$ ) for which  $|[x, y]^{G_0}| = 2$ , then we can write  $y = k \cdot x$  for some  $k \neq 0$ , and  $[x, kx]^h = [x, -kx]$ ,  $[x, -kx]^h = [x, kx]$  which is equivalent to  $(k, 0)^h = (-k, 0)$ ,  $(-k, 0)^h = (k, 0)$ . But  $((-k, 0), (k, 0)) \in T_{2k}$ , whence  $((-k, 0), (k, 0))^h \in T_{2k}$  as well. This implies that  $k = -k$ , a contradiction since  $k \neq 0$  and  $p$  is odd. Hence  $[x, y]$  is fixed by  $G_0$ , and it follows that all points and lines are fixed by  $G_0$ . We conclude that  $|G| = 2p^3$  as desired.
- (d) One easily verifies that the permutation  $\pi$  of part (c) is also an automorphism of  $\mathcal{M}_3$ , i.e.  $\pi \in G := \text{Aut}(\mathcal{M}_3)$ . Moreover, the initial steps of part (c) again establish that  $|[0, 0]^G| = 2p^2$ . We consider once more the stabilizer  $G_0$  of  $(0, 0)$  and  $[0, 0]$ . Because of relations  $T_i$ , the points  $[1, 0], [2, 0], \dots, [p-1, 0]$  are also stabilized by  $G_0$ . Let  $\alpha \in G_0$ . Since  $T_i^\alpha = T_i$ , we have  $[x, y_1]^\alpha = [x, y_2]$  and  $(k, q_1)^\alpha = (k, q_2)$ , i.e.  $\alpha$  does not change the first coordinate of a point or line. In particular,  $\alpha$  preserves each parallel class of lines, hence it permutes the lines  $(0, 1), (0, 2), \dots, (0, p-1)$  amongst themselves. Note that the manner in which  $\alpha$  permutes the lines  $(0, i)$  uniquely determines the images of points under  $\alpha$ . Let  $(1, 0)^\alpha = (c, 0)$  for some  $c \in \mathbb{Z}_p^*$ . Then for all  $x, y \in \mathbb{Z}_p$ , we necessarily have  $[x, y]^\alpha = [x, cy]$ . But  $((0, 0), (1, 0)) \in T_1$ , so  $((0, 0), (1, 0))^\alpha \in T_1$ , i.e.  $((0, 0), (c, 0)) \in T_1$ , which forces  $c = 1$ . Thus  $G_0$  fixes all points as well as lines. It follows that  $|G| = 2p^3$  as desired. (In fact, we confirmed that  $\text{Aut}(\mathcal{M}_2) \cong \text{Aut}(\mathcal{M}_3) \cong \langle t_{01}, t_{10}, \varphi, \pi \rangle$ .)
- (e) Here we set  $G := \text{Aut}(\mathcal{M}_4)$ , and consider the permutations  $\alpha, \beta$  defined by  $[x, y]^\alpha = [x, -y]$ ,  $(k, q)^\alpha = (-k, -q)$  and  $[x, y]^\beta = [-x, -y]$ ,  $(k, q)^\beta = (k, -q)$ . In a fashion similar to parts (b) and (c), one can verify that  $G \cong \langle t_{10}, t_{01}, \varphi, \pi, \alpha, \beta \rangle$ , whence  $|G| = 8p^3$  as claimed.  $\square$

As was mentioned earlier, the results presented in this section were obtained as a theoretical generalisation of a tremendous number of computations, fulfilled with the aid of a computer

for starting small values of prime numbers  $p$ . In addition to the knowledge of combinatorial groups of automorphisms, we were obtaining their ranks and were establishing that, as a rule, the proceeded association schemes are non-Schurian. This, together with Theorem 4, allowed us to formulate our next result.

**Theorem 5** *For  $p > 3$ ,  $\mathcal{M}_1$ ,  $\mathcal{M}_2$ ,  $\mathcal{M}_3$ ,  $\mathcal{M}_4$  are pairwise distinct non-Schurian association schemes.*

**Proof.** Recall our earlier derivation that the number of 2-orbits of  $H = \mathbb{Z}_p^2 \rtimes \mathbb{Z}_p$  on  $V = \mathcal{P} \cup \mathcal{L}$  is  $6p - 2$ . Thus  $\text{Aut}(\mathcal{M}_1)$  is of rank  $6p - 2$ , while the rank of  $\mathcal{M}_1$  is  $3p - 1$ . This proves that  $\mathcal{M}_1$  is non-Schurian for  $p \geq 3$ .

For the association schemes  $\mathcal{M}_2$  and  $\mathcal{M}_3$ , we consider the permutation  $\pi \in \text{Aut}(\mathcal{M}_2) = \text{Aut}(\mathcal{M}_3)$  introduced in part (c) of Theorem 4. As the result of  $\pi$ , we obtain the following 2-orbits:  $A_i \cup C_{-i}$ ,  $B_j \cup D_j$  and  $E_i \cup F_{-i}$ , for  $i \in \mathbb{Z}_p$  and  $j \in \mathbb{Z}_p^*$ . This proves that  $\text{Aut}(\mathcal{M}_2) = \text{Aut}(\mathcal{M}_3)$  is of rank  $3p - 1$ . As the ranks of  $\mathcal{M}_2$  and  $\mathcal{M}_3$  are  $2p$  and  $p + 3$  respectively, we conclude that  $\mathcal{M}_2$  and  $\mathcal{M}_3$  are non-Schurian for  $p \geq 3$ . However, one can check that  $\mathcal{M}_2$  and  $\mathcal{M}_3$  coincide when  $p = 3$ .

Finally, as a result of the permutations  $\alpha, \beta$  introduced in part (e) of Theorem 4, it is easy to see that the 2-orbits of  $\text{Aut}(\mathcal{M}_4)$  are  $A_i \cup A_{-i} \cup C_i \cup C_{-i}$ ,  $B_j \cup B_{-j} \cup D_j \cup D_{-j}$  and  $E_i \cup E_{-i} \cup F_i \cup F_{-i}$ , for  $i \in \{0, 1, \dots, \frac{p-1}{2}\}$  and  $j \in \{1, 2, \dots, \frac{p-1}{2}\}$ . Thus the rank of  $\text{Aut}(\mathcal{M}_4)$  is equal to  $\frac{3p+1}{2}$ . As the rank of  $\mathcal{M}_4$  is  $\frac{p+7}{2}$ , we conclude that  $\mathcal{M}_4$  is non-Schurian for  $p > 3$ .  $\square$

**Remark 4.** Note that when  $p = 3$ , we get only two non-Schurian association schemes, namely  $\mathcal{M}_1$  and  $\mathcal{M}_2$ . Indeed, when  $p = 3$  the association schemes  $\mathcal{M}_2$  and  $\mathcal{M}_3$  coincide, while  $\mathcal{M}_4$  is a Schurian association scheme (here  $\frac{3 \cdot 3 + 1}{2} = \frac{3 + 7}{2} = 5$ ). These two association schemes are of order  $2 \cdot 3^2 = 18$ , and according to the catalogue of small association schemes of Hanaki and Miyamoto [20], we see that our constructions cover all non-Schurian association schemes of order 18.

## 9 A second model of $\mathcal{M}$

According to our intentionally chosen genetic style of presentation, we now consider our favourite objects once again, in a sense, from scratch.

Recall that in the first part of the text our starting object was the biaffine plane  $\mathcal{B}_p$ , and our preferred descriptive language was geometric in nature. Out of necessity, suitable permutations appeared in an ad hoc manner.

In this section we describe a new second model of  $\mathcal{M}$ . (Isomorphism with the first model will be shown later on.) The main advantage of this second model is its purely algebraic flavour. As a result, many of our previous claims will now get more transparent proofs, probably more preferable for readers with developed algebraic tastes.

### 9.1 Initial definitions

We start with description of the second model of  $\mathcal{M}$ .

Let

$$V_1 = \mathbb{Z}_p^2 = \{(1, x_1, x_2) \mid x_1, x_2 \in \mathbb{Z}_p\},$$

$$V_2 = (\mathbb{Z}_p^2)^{dual} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ -1 \end{pmatrix} \mid x_1, x_2 \in \mathbb{Z}_p \right\}.$$

Define a natural scalar product between  $V_1$  and  $V_2$ :

$$(1, x_1, x_2) \begin{pmatrix} y_1 \\ y_2 \\ -1 \end{pmatrix} = y_1 + x_1 y_2 - x_2.$$

Let

$$H' = \left\{ g_{abc} = \begin{pmatrix} 1 & a & b+ac \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{Z}_p \right\}.$$

Matrix  $g_{abc}$  is invertible, and

$$g_{abc}^{-1} = \begin{pmatrix} 1 & -a & -b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}.$$

Clearly, the set  $H'$  together with the operation of matrix-multiplication forms a group. The multiplication in  $H'$  is given by  $g_{abu} \cdot g_{cdv} = g_{a+c, b+d-uc, u+v}$ , and this group is well known under the name *Heisenberg group modulo p*, see e.g. [4].

Define an action of  $H'$  on  $\Omega = V_1 \cup V_2$  by:  $x^g = \begin{cases} x \cdot g & \text{if } x \in V_1 \\ g^{-1} \cdot x & \text{if } x \in V_2, \end{cases}$  for all  $g \in H'$ .

If we take arbitrary  $x_1, x_2, y_1, y_2 \in \mathbb{Z}_p$ , then for  $a = y_1 - x_1$ ,  $b = y_2 - x_2$ ,  $c = 0$  matrix  $g_{abc}$  sends  $(1, x_1, x_2) \in V_1$  to  $(1, y_1, y_2) \in V_1$ , while  $g_{cab}$  sends  $(x_1, x_2, -1)^T \in V_2$  to  $(y_1, y_2, -1)^T \in V_2$ , therefore  $H'$  acts transitively on  $V_1$  and  $V_2$  as well.

The action of  $H'$  preserves the scalar product:  $x^g \cdot y^g = (x \cdot g) \cdot (g^{-1} \cdot y) = x \cdot gg^{-1} \cdot y = x \cdot y$ .

**Proposition 6** *Groups  $H$  and  $H'$  are isomorphic.*

**Proof.** We claim that  $\Phi : H \rightarrow H'$ ,  $h_{a,b,u} \mapsto g_{a,b+au,-u}$  is a group isomorphism. First, we have

$$\begin{aligned} \Phi(h_{a,b,u} \circ h_{c,d,v}) &= \Phi(h_{a+c, b+d-av, u+v}) = g_{a+c, b+d-av+(a+c)(u+v), -(u+v)} = \\ &= g_{a, b+au, -u} \cdot g_{c, d+cv, -v} = \Phi(h_{a,b,u}) \cdot \Phi(h_{c,d,v}). \end{aligned}$$

Moreover,  $\Phi$  is invertible:  $\Phi^{-1}(g_{\alpha, \beta, \gamma}) = h_{\alpha, \beta+\alpha\gamma, -\gamma}$ . Thus,  $\Phi$  is a group isomorphism from  $H$  to  $H'$  as claimed.  $\square$

Proposition 6 allows us to identify groups  $H$  and  $H'$ , which is why we shall henceforth ascribe the notation  $H$  to both groups. However, the reader is advised to keep in mind the group  $H'$  as it appears in this section.

## 9.2 Orbits of $H$ on $\Omega^2$

The isomorphism between groups  $H$  and  $H'$  established in Proposition 6 allows an alternate description of the 2-orbits of the biaffine coherent configuration  $\mathcal{M}$ , and correspondingly, an alternate proof of Proposition 1.

Not attempting to put such a proof into the framework of formal propositions, we nevertheless outline the necessary arguments and final formulations for the reader's benefit.

### Orbits on $V_1 \times V_1$ :

Let  $g = g_{abc} \in H$  and  $P_1 = (1, x_1, x_2)$ ,  $P_2 = (1, y_1, y_2)$ ,  $P_3 = (1, u_1, u_2)$ ,  $P_4 = (1, v_1, v_2) \in V_1$ . Then  $(P_1, P_2)^g = (P_3, P_4)$  if and only if

$$\begin{aligned} u_1 &= x_1 + a \\ v_1 &= y_1 + a \\ u_2 &= c \cdot x_1 + x_2 + b + a \cdot c \\ v_2 &= c \cdot y_1 + y_2 + b + a \cdot c. \end{aligned}$$

We can see that if  $(P_1, P_2)$  and  $(P_3, P_4)$  belong to the same orbit, then necessarily  $y_1 - x_1 = v_1 - u_1$ ,  $a = u_1 - x_1 = v_1 - y_1$ , and  $v_2 - u_2 = c(y_1 - x_1) + (y_2 - x_2)$ .

- (i) If  $y_1 - x_1 = 0$ , then it is necessary to have  $v_2 - u_2 = y_2 - x_2$ , and by choosing  $c = 0$  and  $b = u_2 - x_2$  we are getting a suitable  $g$  sending  $(P_1, P_2)$  to  $(P_3, P_4)$ .
- (ii) If  $y_1 - x_1 = k \in \mathbb{Z}_p^*$ , then we can choose  $c = k^{-1}(v_2 - u_2 + x_2 - y_2)$ ,  $b = u_2 - cx_1 - x_2 - ac$  and we get  $(P_1, P_2)^g = (P_3, P_4)$ .

Hence we have got two types of orbits, say  $A_k$  and  $B_k$  on  $V_1 \times V_1$ :

- $(P_1, P_2) \in A_k$  if and only if  $y_1 = x_1$  and  $y_2 - x_2 = k$ , where  $k \in \mathbb{Z}_p$ .
- $(P_1, P_2) \in B_k$  if and only if  $y_1 - x_1 = k$ , where  $k \in \mathbb{Z}_p^*$ .

### Orbits on $V_2 \times V_2$ :

Let  $g = g_{abc} \in H$  and  $l_1 = (x_1, x_2, -1)^T$ ,  $l_2 = (y_1, y_2, -1)^T$ ,  $l_3 = (u_1, u_2, -1)^T$ ,  $l_4 = (v_1, v_2, -1)^T \in V_2$ . Then  $(l_1, l_2)^g = (l_3, l_4)$  if and only if

$$\begin{aligned} u_1 &= x_1 - ax_2 + b \\ v_1 &= y_1 - ay_2 \\ u_2 &= x_2 + c \\ v_2 &= y_2 + c. \end{aligned}$$

We can see that if  $(l_1, l_2)$  and  $(l_3, l_4)$  belong to the same orbit, then necessarily  $y_2 - x_2 = v_2 - u_2$ ,  $c = u_2 - x_2 = v_2 - y_2$ , and  $v_1 - u_1 = a(x_2 - y_2) + (y_1 - x_1)$ .

- (i) If  $y_2 - x_2 = 0$ , then it is necessary to have  $v_1 - u_1 = y_1 - x_1$ , and by choosing  $a = 0$  and  $b = v_1 - y_1$  we obtain a suitable  $g$  sending  $(l_1, l_2)$  to  $(l_3, l_4)$ .
- (ii) If  $x_2 - y_2 = k \in \mathbb{Z}_p^*$ , then we can choose  $a = k^{-1}(v_1 - u_1 + x_1 - y_1)$ ,  $b = v_1 - y_1 + ay_2$  and we get  $(l_1, l_2)^g = (l_3, l_4)$ .

Hence we get two types of orbits, say  $C_k$  and  $D_k$ , on  $V_2 \times V_2$ :

- $(l_1, l_2) \in C_k$  if and only if  $y_2 = x_2$  and  $y_1 - x_1 = k$ , where  $k \in \mathbb{Z}_p$ .
- $(l_1, l_2) \in D_k$  if and only if  $y_2 - x_2 = k$ , where  $k \in \mathbb{Z}_p^*$ .

**Orbits on  $V_1 \times V_2$ :**

Let  $P_1 = (1, x_1, x_2)$ ,  $P_2 = (1, y_1, y_2) \in V_1$  and  $l_1 = (u_1, u_2, -1)^T$ ,  $l_2 = (v_1, v_2, -1)^T \in V_2$ .

Since our scalar product is preserved by  $H$ , it follows that  $(P_1, l_1)$  and  $(P_2, l_2)$  belong to different orbits when  $P_1 \cdot l_1 \neq P_2 \cdot l_2$ . Now suppose  $P_1 \cdot l_1 = P_2 \cdot l_2$ , i.e.

$$y_2 = v_1 + y_1 v_2 + x_2 - u_1 - x_1 u_2. \quad (1)$$

We will show that in this case there exists  $g \in H$  such that  $(P_1, l_1)^g = (P_2, l_2)$ . Necessarily,

$$\begin{aligned} y_1 &= x_1 + a \\ y_2 &= x_2 + c x_1 + b + a c \\ v_1 &= u_1 - a u_2 + b \\ v_2 &= u_2 + c. \end{aligned}$$

Choosing  $a = y_1 - x_1$ ,  $c = v_2 - u_2$ ,  $b = v_1 - u_1 + a u_2$ , all equations are fulfilled and therefore  $P_1^g = P_2$ ,  $l_1^g = l_2$ . Thus the orbits  $E_k$  of  $H$  on  $V_1 \times V_2$  are formed by pairs with the same value of scalar product:  $(P, l) \in E_k \iff P \cdot l = k$ , where  $k \in \mathbb{Z}_p$ .

**Orbits on  $V_2 \times V_1$ :**

Similar to the previous case, we obtain orbits  $F_k$ :  $(l, P) \in F_k \iff P \cdot l = -k$ .

Altogether, we get the same six types of orbits of  $H$  on  $\Omega \times \Omega$  with which we are already familiar:

$$\begin{aligned} (P_1, P_2) \in A_k &\iff y_1 = x_1 \text{ and } y_2 - x_2 = k \in \mathbb{Z}_p \\ (P_1, P_2) \in B_k &\iff y_1 - x_1 = k \in \mathbb{Z}_p^* \\ (l_1, l_2) \in C_k &\iff u_2 = v_2 \text{ and } v_1 - u_1 = k \in \mathbb{Z}_p \\ (l_1, l_2) \in D_k &\iff v_2 - u_2 = k \in \mathbb{Z}_p^* \\ (P_1, l_1) \in E_k &\iff P_1 \cdot l_1 = k \in \mathbb{Z}_p \\ (l_1, P_1) \in F_k &\iff P_1 \cdot l_1 = -k \in \mathbb{Z}_p, \end{aligned}$$

where  $P_1 = (1, x_1, x_2)$ ,  $P_2 = (1, y_1, y_2) \in V_1$  and  $l_1 = (u_1, u_2, -1)^T$ ,  $l_2 = (v_1, v_2, -1)^T \in V_2$ .

## 10 Toward a theoretical understanding of the algebraic groups for the detected coherent configurations

At this stage we are ready to discuss how, in principle, one may describe the full algebraic group of a prescribed coherent configuration  $\mathcal{W}$ , in particular, of an association scheme.

Recall that  $\text{AAut}(\mathcal{W})$  acts faithfully on the set of relations from  $\mathcal{W}$ , preserving the tensor  $\mathcal{T}$  of structure constants of  $\mathcal{W}$ .

Roughly speaking, our methodology may be described as follows:



- detect a few permutations, say  $\varphi_1, \varphi_2, \dots, \varphi_k$ , acting on the set of relations of  $\mathcal{W}$ ;
- check that each permutation  $\varphi_i$ ,  $1 \leq i \leq k$ , preserves  $\mathcal{T}$ ;
- describe group  $K = \langle \varphi_1, \varphi_2, \dots, \varphi_k \rangle$  as abstract group, and establish its orbits on the set of relations of  $\mathcal{R}$ ;
- prove, using ad hoc arguments, that each element  $\varphi \in \text{AAut}(\mathcal{W})$  may be expressed as a suitable concatenation  $x_1 x_2 \dots x_l$ , where  $x_j \in \{\varphi_1, \varphi_2, \dots, \varphi_k\}$  for  $1 \leq j \leq l$ .

Note that in general, the final step seems quite sophisticated. Nevertheless, for small numbers  $k$  of generators, the tricks we employed appear to be fairly visible and reasonably natural.

There is a definite sense to stress that the problem of calculating  $\text{AAut}(\mathcal{W})$  is a relatively new kind of activity in AGT. The paper [26] was among the first in which such types of reasonings were considered. In a number of other publications the calculation was achieved with the aid of a computer. One of the foremost goals of the current text is to introduce some helpful tricks and theoretical reasonings which allow one, in principle, to attack the problem with a sufficient level of rigour. Thus, we will first demonstrate the suggested methodology on the association scheme  $\mathcal{M}_1$ .

**Proposition 7** *Permutations*

$$\begin{aligned}\alpha &= (T_1, T_\omega, T_{\omega^2}, \dots, T_{\omega^{p-2}}), \\ \beta &= (S_1, S_\omega, S_{\omega^2}, \dots, S_{\omega^{p-2}})(U_1, U_\omega, U_{\omega^2}, \dots, U_{\omega^{p-2}}),\end{aligned}$$

acting on the set of relations of the association scheme  $\mathcal{M}_1$ , preserve its tensor  $\mathcal{T}_1$  of structure constants. Here  $\omega$  is a primitive element of  $\mathbb{Z}_p$  regarded as a field (in other words,  $\omega$  is a generator of the multiplicative group  $\mathbb{Z}_p^*$ ).

**Proof.** We refer to the Appendix 1, where the tensor  $\mathcal{T}_1$  is presented with the aid of four tables. Permutation  $\alpha$  acts only on the set of relations of type  $T_i$  as  $T_i^\alpha = T_{i\omega}$ , therefore it is enough to check what is happening with the intersection numbers related to this type of relations. For example,

$$c_{ti,tj}^0 = p \cdot \delta_{i,j'}, \quad c_{ti^\alpha,tj^\alpha}^0 = c_{ti\omega,tj\omega}^0 = p \cdot \delta_{i\omega,(j\omega)'}$$

So we need to show that  $\delta_{i,j'} = \delta_{i\omega,(j\omega)'}$ . Our relations satisfy  $i' = -i$ , hence the equation  $i = -j$  is equivalent to  $i\omega = -j\omega$ , in which case we are done. In the case of intersection numbers of the type  $c_{ti,tj}^{tk}$  we need to show that  $\delta_{i+j,k} = \delta_{i\omega+j\omega,k\omega}$ , which is obvious. Using similar reasonings, one proves that  $\beta$  preserves  $\mathcal{T}_1$  as well. □

We now observe that both  $\alpha$  and  $\beta$  are cyclic permutations of order  $p-1$ . Also, we easily check that  $\alpha$  and  $\beta$  commute by making a straightforward comparison of  $\alpha\beta$  with  $\beta\alpha$ . This implies that  $\langle \alpha, \beta \rangle \cong \mathbb{Z}_{p-1}^2$ . Denote by  $K$  this group  $\mathbb{Z}_{p-1}^2$  in its action on the relations of  $\mathcal{M}_1$ .

**Corollary 8**  $\text{AAut}(\mathcal{M}_1) \geq K$ .

Our next step is simply to prove that each  $\varphi \in \text{AAut}(\mathcal{M}_1)$  indeed belongs to  $K$ .

**Theorem 9**  $\text{AAut}(\mathcal{M}_1) \cong \mathbb{Z}_{p-1}^2$ .

**Proof.** Let  $\varphi \in \text{AAut}(\mathcal{M}_1)$ . The only relation  $i$  in  $\mathcal{M}_1$  for which  $c_{ii}^i = 1$  is  $i = R_0$ , therefore  $R_0^\varphi = R_0$ . Since  $c_{si,si'}^0 = 1$  and there are no other 1's among the values of type  $c^0$ , we have that  $S_i^\varphi = S_j$  for some  $j$ . Moreover,  $S_{i'}^\varphi = S_{j'}$ . Just using values of type  $c^0$ , similar considerations lead to the following observations: for all  $i$  there exists  $j$  such that  $T_i^\varphi = T_j$ , and for all  $l$  there exists  $m$  such that  $U_l^\varphi = U_m$ . In particular,  $U_0^\varphi = U_0$ .

Now let  $S_1^\varphi = S_\omega$ , where  $\omega$  is a primitive element of  $\mathbb{Z}_p^*$ . Then by induction on  $i$ ,  $c_{s1,si}^{si+1} = 1$  implies  $S_{i+1}^\varphi = S_{\omega(i+1)}$ . Similarly  $T_i^\varphi = T_\omega$  implies  $T_i^\varphi = T_{\omega \cdot i}$ . However  $S_i^\varphi = S_{\omega \cdot i}$ , and  $c_{u0,ui}^{si} = p = c_{u0,u,j}^{s\omega i}$ , therefore  $j = \omega i$ , i.e.,  $U_i^\varphi = U_{\omega i}$ .

Letting  $\omega$  be a primitive element of  $\mathbb{Z}_p$ , the above computations establish that  $\text{AAut}(\mathcal{M}_1) \leq \langle \alpha, \beta \rangle$ . Together with Corollary 8, we conclude have  $\text{AAut}(\mathcal{M}_1) = \langle \alpha, \beta \rangle$  from which follows  $\text{AAut}(\mathcal{M}_1) \cong \mathbb{Z}_{p-1}^2$ .  $\square$

## 11 Algebraic group of the coherent configuration $\mathcal{M}$

In the previous section we gave a description of the group  $\text{AAut}(\mathcal{M}_1)$ . We hope that our presentation fulfilled its mission: to create for the reader a first acquaintance with this significant concept.

In this section we will describe the group  $\text{AAut}(\mathcal{M})$ . Knowledge of the group will play a critical role in achieving our further goals. Once more we wish to stress significant features of the used technology. Indeed, at the first stage the description of the order of  $\text{AAut}(\mathcal{M})$  was obtained for initial values of the prime parameter  $p$ . Again aided by GAP, we next made educated guesses at the structure of these groups for the obtained orders. Finally, an evident formulation of a theorem became transparent to us. From this we created a pedestrian proof, a brief outline of which is given below.

Thus let us start with the following permutations on the set of relations of the coherent configuration  $\mathcal{M}$ . (We assume  $\omega$  is a primitive element of  $\mathbb{Z}_p^*$ .)

$$\begin{aligned} g_1 &= (A_0, C_0)(E_0, F_0) \prod_{i=1}^{p-1} (A_i, C_i)(B_i, D_i)(E_i, F_i), \\ g_2 &= (E_0, E_{p-1}, E_{p-2}, \dots, E_2, E_1)(F_0, F_1, F_2, \dots, F_{p-1}), \\ g_3 &= (A_1, A_\omega, A_{\omega^2}, \dots, A_{\omega^{p-2}})(C_1, C_\omega, \dots, C_{\omega^{p-2}})(E_1, E_\omega, \dots, E_{\omega^{p-2}})(F_1, F_\omega, \dots, F_{\omega^{p-2}}), \\ g_4 &= (B_1, B_\omega, B_{\omega^2}, \dots, B_{\omega^{p-2}}), \\ g_5 &= (D_1, D_\omega, D_{\omega^2}, \dots, D_{\omega^{p-2}}). \end{aligned}$$

Permutation  $g_1$  is an involution which corresponds to the duality between points and lines in  $\mathcal{B}_p$  (it interchanges the roles of points and lines). Permutation  $g_2$  is of order  $p$ , while the remaining permutations are of order  $p-1$ .

**Theorem 10** *The group  $\text{AAut}(\mathcal{M})$  is of order  $2p(p-1)^3$  and*

$$\text{AAut}(\mathcal{M}) \cong \langle g_1, g_2, g_3, g_4, g_5 \rangle \cong (\mathbb{Z}_{p-1}^2 \rtimes \mathbb{Z}_2) \times \text{AGL}(1, p).$$

**Proof.** First, we have to check that each of the permutations  $g_1, \dots, g_5$  preserves the tensor of structure constants. This job can be done by hand, although we admit it is a bit tedious. Nevertheless, we suggest that inspection of at least one of the presented permutations may serve as a helpful exercise for the reader. Thus, we have that  $\langle g_1, g_2, g_3, g_4, g_5 \rangle \leq \text{AAut}(\mathcal{M})$ . In order to see that  $\text{AAut}(\mathcal{M}) \cong \langle g_1, g_2, g_3, g_4, g_5 \rangle$  it is sufficient to compute and compare the orders of  $\langle g_1, g_2, g_3, g_4, g_5 \rangle$  and  $\text{AAut}(\mathcal{M})$ . The order of  $\text{AAut}(\mathcal{M})$  may be determined by repeated application of the orbit-stabilizer lemma. The order of  $\langle g_1, g_2, g_3, g_4, g_5 \rangle$  will be derived by us presently.

Clearly  $\{g_2, g_3\}$  is a standard set of generators for the affine linear group over the finite field  $\mathbb{F}_p$  of order  $p$ , that is,  $\langle g_2, g_3 \rangle \cong \text{AGL}(1, p)$ . Also it is easy to see that  $\langle g_4, g_5 \rangle \cong \mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1} = \mathbb{Z}_{p-1}^2$  and  $\langle g_1, g_4, g_5 \rangle \cong \mathbb{Z}_{p-1}^2 \rtimes_{\phi} \mathbb{Z}_2$ , where  $\phi$  interchanges  $g_4$  with  $g_5$ . By routine computation one can further show that  $\langle g_1, g_4, g_5 \rangle \trianglelefteq \langle g_1, g_2, g_3, g_4, g_5 \rangle$ ,  $\langle g_2, g_3 \rangle \trianglelefteq \langle g_1, g_2, g_3, g_4, g_5 \rangle$ , and  $\langle g_1, g_4, g_5 \rangle \cap \langle g_2, g_3 \rangle = \{e\}$ . Thus  $\langle g_1, g_4, g_5 \rangle \times \langle g_2, g_3 \rangle \cong \langle g_1, g_2, g_3, g_4, g_5 \rangle$ , which confirms that  $\langle g_1, g_2, g_3, g_4, g_5 \rangle \cong (\mathbb{Z}_{p-1}^2 \rtimes \mathbb{Z}_2) \times \text{AGL}(1, p)$ . Finally,  $|\langle g_1, g_2, g_3, g_4, g_5 \rangle| = 2p(p-1)^3$ .  $\square$

## 12 Detected non-Schurian association schemes as algebraic mergings

As we have seen in the previous section, the algebraic group  $\text{AAut}(\mathcal{M})$  is of order  $2p(p-1)^3$ . It is easy to check that this group has four orbits of length 2,  $2p-2$ ,  $2p-2$  and  $2p$ , respectively, on the set of relations of  $\mathcal{M}$ . These orbits are:

$$\{A_0, C_0\}, \{A_1, \dots, A_{p-1}, C_1, \dots, C_{p-1}\}, \{B_1, \dots, B_{p-1}, D_1, \dots, D_{p-1}\}, \\ \{E_0, E_1, \dots, E_{p-1}, F_0, F_1, \dots, F_{p-1}\}.$$

Knowledge of the algebraic group of automorphisms of a coherent configuration is important for constructing algebraic mergings. It is known (see [26]) that if we take the orbits of a subgroup  $H$  of  $\text{AAut}(\mathcal{W})$ , then the algebraic merging with respect to the partition of the relations into orbits of  $H$ , leads to a coherent configuration. In particular, if  $\mathcal{W}$  is an association scheme, then its algebraic merging is also an association scheme.

### 12.1 New interpretation of the main non-Schurian association schemes

Here we present one more result, which was initially obtained via plausible reasonings based on observation of computer aided data for diverse values of  $p$ . Its essential feature is that the proof becomes almost trivial, provided our suggested methodology of the use of algebraic groups is exploited in the correct manner.

**Proposition 11** *The association schemes  $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ , and  $\mathcal{M}_4$  appear as algebraic mergings of  $\mathcal{M}$ .*

**Proof.** Let  $q = (p-1)/2$ . Then the algebraic mergings corresponding to the subgroups  $K_1 = \langle g_1 \rangle$ ,  $K_2 = \langle g_1, g_3^q \rangle$ ,  $K_3 = \langle g_1, g_3 \rangle$  and  $K_4 = \langle g_1, g_3, g_4^q, g_5^q \rangle$  of  $\text{AAut}(\mathcal{M})$  lead to the association schemes  $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$  and  $\mathcal{M}_4$ , respectively.

□

This proposition provides an alternate proof that the color graphs defined in Section 6 correspond to association schemes. Recall that their ranks are  $3p - 1$ ,  $2p$ ,  $p + 3$  and  $(p + 7)/2$ , respectively.

We are already aware that for  $p > 3$  these algebraic mergings are non-Schurian.

## 12.2 Experimental results with color and algebraic automorphisms

Computer experiments performed over a sufficiently large interval of initial values of the prime parameter  $p$  show that  $|\text{CAut}(\mathcal{M})| = 2p^4(p - 1)^2$ . This implies that the order of the quotient group  $\text{CAut}(\mathcal{M})/\text{Aut}(\mathcal{M})$  is  $\frac{2p^4(p-1)^2}{p^3} = 2p(p - 1)^2$ . Comparing this with the order of the group  $\text{AAut}(\mathcal{M})$ , we see that the index of the quotient group in  $\text{AAut}(\mathcal{M})$  is  $p - 1$ . Thus for all  $p \geq 3$  there exist proper algebraic mergings.

It was at this particular stage of our project that we became greatly enthused. Indeed, we fully realized that the existence of proper algebraic mergings would create for us a most attractive research agenda that could potentially guide us to promising new discoveries.

In particular, for a few initial values of  $p$  we arranged a full search of subgroups of  $\text{AAut}$ , constructions of corresponding mergings, and an investigation of the properties of the resulting coherent configurations and association schemes.

At this stage we are not yet prepared to transform our observations into the rigorous platform of proved mathematical claims. Indeed, our intentions are to do this in forthcoming publications (see discussion at the end of the text). Nevertheless, a portion of them will be considered in more detail in Section 14.

## 13 Links to known combinatorial structures

In this section we mention a few well known combinatorial structures, especially graphs, which are somehow related to the color graphs we have explored in this paper.

The graph defined by color  $U_0$  for  $p = 3$  goes by the name *Pappus graph*. It is the incidence graph of the *Pappus configuration* (see e.g. [11]).

The so-called *McKay-Miller-Širáň graphs*  $H_p$  are well known in the degree/diameter problem (see [34] for a survey) which falls within the realm of extremal graph theory. They were defined originally in [33], later on Šiagiová [41] and finally Hafner [19] gave simplified constructions. We will define them according to Hafner's description. Though he described these graphs in terms of biaffine planes for arbitrary prime powers, for our purposes it suffices to restrict our attention to the case of odd primes. So let  $p$  be an odd prime and put  $V_p = \mathbb{Z}_2 \times \mathbb{Z}_p \times \mathbb{Z}_p$  as the vertex set of  $H_p$ . Let  $\omega$  be a primitive element of  $\mathbb{Z}_p^*$ . Since  $p$  is an odd prime, there are two possibilities:

- If  $p = 4r + 1$  then define  $X = \{1, \omega^2, \omega^4, \dots, \omega^{p-3}\}$ ,  $X' = \{\omega, \omega^3, \dots, \omega^{p-2}\}$ .
- If  $p = 4r + 3$  then define  $X = \{\pm 1, \pm \omega^2, \dots, \pm \omega^{2r}\}$ ,  $X' = \{\pm \omega, \pm \omega^3, \dots, \pm \omega^{2r+1}\}$ .

Adjacency in  $H_p$  is defined as follows:

$$\begin{aligned} (0, x, y) \text{ is adjacent to } (0, x, y') & \quad \text{if and only if} \quad y - y' \in X, \\ (1, k, q) \text{ is adjacent to } (1, k, q') & \quad \text{if and only if} \quad q - q' \in X', \\ (0, x, y) \text{ is adjacent to } (1, k, q) & \quad \text{if and only if} \quad y = kx + q. \end{aligned}$$

From this description it is clear that McKay-Miller-Širáň graphs may be obtained as a suitable merging of relations of  $\mathcal{M}$ . Specifically:

$$H_p = E_0 \cup F_0 \cup \bigcup_{i \in X} A_i \cup \bigcup_{j \in X'} C_j.$$

In particular,  $H_5$  is the well known Hoffman-Singleton graph [23].

The McKay-Miller-Širáň graphs are currently the best known solutions to the degree/diameter problem for diameter 2 and valency  $(3q - 1)/2$ , where  $q$  is a prime power. They still play a significant role in newer constructions that give denser graphs, i.e. graphs that are closer to the Moore bound.

Recently, motivated by the success of McKay-Miller-Širáň graphs, authors in [1] investigated how to extend the graph corresponding to  $U_0$  in order to find better constructions in the degree/diameter problem. We plan to consider this operation of extension with more details in the future, because it fits quite well the language of relations in  $\mathcal{M}$  developed in our paper.

Another family of graphs which may be defined in our terminology is the family of *Wenger graphs*  $W_1(p)$  introduced in [46]. They have  $2p^2$  vertices with edge set coinciding with the relation  $U_0 = E_0 \cup F_0$ . In other words, they are flag graphs of the biaffine plane. The Wenger graphs were studied for their extremal properties and belong to a richer family of graphs defined by systems of equations on the coordinates of points and lines. For more details, see [29, 30, 44, 48].

## 14 Some association schemes of small rank, appearing as mergings of $\mathcal{M}$

Recall that the main result of this paper is the discovery and investigation of four infinite families of non-Schurian association schemes, which arise as mergings of the coherent configuration  $\mathcal{M}$  defined on  $2p^2$  elements of the classical biaffine plane of odd prime order  $p$ . For each of these four families, rank of the schemes grows with increasing  $p$ .

From the earliest inception of AGT, special attention has been paid to association schemes of small rank. The smallest possible (non-trivial) rank is 3, which corresponds to strongly regular graphs in the symmetric case, and doubly regular tournaments in the non-symmetric case. Applying special efforts, one can attempt to prove that in our case, for  $p > 5$ , such primitive objects cannot appear. Nevertheless, computer experiments created for us evidence that already for small values of  $p$  we are getting non-Schurian mergings of constant low ranks 5 and 6. This immediately created a new challenge for us: how to justify existence of such schemes for arbitrary values of  $p$ .

The corresponding results are more fresh, with a portion of them having been announced by the author M.K. at the conference “Modern trends in AGT” held at Villanova University in June 2014. (See Section 15.9 for a link to the slides of this announcement.)

In this section we provide an outline of the report of these results. It is given in the form of research announcement, that is, we are not aiming to give a full formulation or justification of our claims.

## 14.1 Starting rank 6 scheme

We begin with a brief discussion of our technology, which we feel may hold independent interest due to its innovative combination of computation and further reasonings.

Recall that for a few small values of  $p$  a full enumeration of all coherent subalgebras was obtained (see Section 15). Further analysis of these results suggested to us that it might be possible to obtain a non-Schurian merging association scheme of rank 6 for every value of  $p$ . Moreover, we found evidence to support that such a scheme might appear as a suitable algebraic merging. This led us to consider the subgroup  $K$  of  $\text{AAut}(\mathcal{M})$ , where  $K = \langle g_1, g_3, g_4^2, g_5^2 \rangle$ . (Here we are following the notation introduced in Section 11.)

Our next step was to apply our algebraic group  $K$  to  $\mathcal{M}$  for all odd primes  $p \leq 19$ . In each case we constructed the corresponding algebraically merged association scheme and investigated its main properties. For the reader's convenience a summary of our results is presented in Appendix 2.

A more careful theoretical analysis of our observations allowed us to reach the general picture for arbitrary primes  $p$ .

**Announcement 1.** For all odd primes  $p$  there exists a non-Schurian rank 6 algebraic merging  $\mathcal{N}_6$  of the master coherent configuration  $\mathcal{M}$ . The group  $\text{Aut}(\mathcal{N}_6)$  is a transitive rank 8 group of order  $\frac{1}{2}(p-1)^2 p^3$ . The group  $\text{CAut}(\mathcal{N}_6)$  has twice larger order. The group  $\text{AAut}(\mathcal{N}_6)$  has order 2 and thus coincides with the group  $\text{CAut}(\mathcal{N}_6)/\text{Aut}(\mathcal{N}_6)$ . The scheme  $\mathcal{N}_6$  is commutative. It is non-symmetric when  $p \equiv 3 \pmod{4}$  and symmetric when  $p \equiv 1 \pmod{4}$ .

At the next stage we were able to describe the full tensor of structure constants of the scheme  $\mathcal{N}_6$ . The results depend on the modulo 4 congruence class of  $p$ , hence they are presented as two separate cases in the slides of M.K. mentioned above.

Moreover, based on the use of classical techniques in AGT and relying on the knowledge of structure constants, we were able to describe the characteristic polynomials and spectra of the basic graphs of  $\mathcal{N}_6$ .

**Announcement 2.** Denote by  $\Lambda_i$  the spectrum of the basic graph  $\mathcal{N}_{6,i}$  in  $\mathcal{N}_6$ . Then:

$$\begin{aligned} \Lambda_1 &= \{1^{2p^2}\} \\ \Lambda_2 &= \{p-1^{2p}, -1^{2p^2-2p}\} \\ \Lambda_3 = \Lambda_4 &= \begin{cases} \{\frac{p(p-1)^2}{2}, 0^{2p^2-2p}, \frac{p}{2}(-1-\sqrt{5})^{p-1}, \frac{p}{2}(-1+\sqrt{5})^{p-1}\} & \text{if } p \equiv 1 \pmod{4} \\ \{\frac{p(p-1)^2}{2}, 0^{2p^2-2p}, \frac{p}{2}(-1-i\sqrt{3})^{p-1}, \frac{p}{2}(-1+i\sqrt{3})^{p-1}\} & \text{if } p \equiv 3 \pmod{4} \end{cases} \\ \Lambda_5 &= \{p^1, 0^{2p-2}, -p^1, -\sqrt{p}^{p(p-1)}, \sqrt{p}^{p(p-1)}\} \\ \Lambda_6 &= \{p(p-1)^1, 0^{2p-2}, -p(p-1)^1, -\sqrt{p}^{p(p-1)}, \sqrt{p}^{p(p-1)}\}. \end{aligned}$$

As is customary, we use superscripts to indicate the multiplicity of each given eigenvalue.

**Remark 5.** It is worth mentioning that  $\text{Aut}(\mathcal{N}_6)$  has constant rank 8. Due to this, all mergings of  $\text{Aut}(\mathcal{N}_6)$  might be described with the aid of COCO, at least for all values  $p < 100$ . Though this was accomplished by us for only a few small values of  $p$  (in parallel with computations in GAP), such an approach might be helpful in conducting a more careful future analysis of similarly obtained objects of constant rank.

## 14.2 Rank 5 mergings

It turns out that the Schurian rank 8 association scheme, which appears from the 2-orbits of  $\text{Aut}(\mathcal{N}_6)$  has two non-Schurian rank 5 mergings. Relevant information for one of these schemes, denoted  $\mathcal{N}_{5,1}$ , is presented in the slides of M.K. Here, we shall restrict our attention only to consideration of the spectrum of  $\mathcal{N}_{5,1}$ . We believe it is of an independent interest due to its more sophisticated structure.

**Announcement 3.** Denote again by  $\Lambda_i$  the spectrum of the basic graph  $\mathcal{N}_{5,1,i}$  in  $\mathcal{N}_{5,1}$ . Then

$$\begin{aligned}\Lambda_1 &= \{1^{2p^2}\} \\ \Lambda_2 &= \{p - 1^{2p}, -1^{2p^2-2p}\} \\ \Lambda_3 &= \{-p^{2p-2}, 0^{2p^2-2p}, p(p-1)^2\} \\ \Lambda_4 &= \left\{ \pm \frac{p(p-1)^1}{2}, 0^{2p-2}, -\frac{1}{2}\sqrt{p(p+1)}^{p(p-1)}, \frac{1}{2}\sqrt{p(p+1)}^{p(p-1)} \right\} \\ \Lambda_5 &= \left\{ \pm \frac{p(p+1)^1}{2}, 0^{2p-2}, -\frac{1}{2}\sqrt{p(p+1)}^{p(p-1)}, \frac{1}{2}\sqrt{p(p+1)}^{p(p-1)} \right\}\end{aligned}$$

As before, superscripts are used to indicate multiplicities of eigenvalues.

## 14.3 Extra discussion

Use of the term “announcement” in this section was a conscious decision on our part, intended to stress the following:

- In the beginning, all results were obtained through plausible reasonings based on a careful analysis of numerous computer algebra experiments;
- We are aware of all necessary tools in order to transform the results of plausible reasonings to rigourously justified theoretical propositions;
- Some such justifications have already been reached, while others still require further effort and are postponed to the future;
- A full justification of the formulated results does not fit into the agenda of this paper, as much more will be needed in the way of preliminaries and discussions of used techniques;
- We expect to revisit this topic in the future and to devote to it a separate new paper;
- This section intends also to underline our priority, at least in the formulation of all the communicated results.

One of the essential features of the planned continuation of the presented research is working on the edge between AGT and extremal graph theory (briefly EGT). The established methodology for determining spectra of basic graphs is fairly traditional in the theory of association schemes, while tools exploited in EGT are of a quite different nature. This is why we hope in our future work to present some fresh vision of a few classes of graphs exploited in EGT. There are also expectations that more careful analysis of some graphs in our schemes, in particular those of constant rank, may imply innovative results in the sphere of EGT.

## 15 Concluding discussion

### 15.1 Origins of the project

This research started in the framework of postdoctoral studies of the author Š.Gy. at the Ben-Gurion University of the Negev, beginning in November 2011. During the initial stages, Š.Gy. was introduced to the main concepts of AGT and was garnering new experience in the use of a number of computer packages, as described in Section 3.

In particular, it was suggested to attempt to understand without the use of a computer, the structure of two non-Schurian association schemes on 18 points, which had been known for a long time.

Quite soon such a computer-free interpretation (strictly in terms of [27]) was elaborated. Moreover, it became clear that there was space for clever generalisations. Further natural generalisations were elaborated, leading eventually to an understanding of the first model of the coherent configuration  $\mathcal{M}$  and its four mergings.

At the next stage, ideas developed by F. Lazebnik, V. Ustimenko and A.J. Woldar, with which the author M.K. was already acquainted, were successfully exploited to understand the potential of the second model. Moreover, both authors were sharing mutual pleasure from the ongoing feeling that the techniques of algebraic mergings outlined in [26] works perfectly well in our case also.

Though this research was arranged without specific use of the techniques discussed in [48], we believe it is quite likely that there is a very promising potential impact from the amalgamation of the two approaches.

### 15.2 More about some other computations

While we think we have already devoted sufficient attention to diverse computer aided activities in previous sections, we nonetheless feel compelled to at least briefly discuss a couple of extra approaches not touched upon earlier.

All computations considered in these subsections were executed with the aid of COCO II.

#### 15.2.1 Algebraic stabilizers

The group  $\text{AAut}(\mathcal{W})$  is a permutation group acting on the set of colors of  $\mathcal{W}$ . Each merging corresponds to a partition  $P$  of the set of colors of  $\mathcal{W}$ . In effect, it is a set of sets of colors. If  $P$  is an algebraic merging then it consists of the orbits of some subgroup of  $\text{AAut}(\mathcal{W})$ . We can ask what is the *algebraic stabilizer* of a partition  $P$ , i.e. the largest subgroup of  $\text{AAut}(\mathcal{W})$  which leaves  $P$  invariant.

The algebraic stabilizers of association schemes  $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$  and  $\mathcal{M}_4$  are isomorphic to  $\mathbb{Z}_2$ ,  $\mathbb{Z}_2 \times \mathbb{Z}_2$ ,  $\mathbb{Z}_{p-1} \times \mathbb{Z}_2$  and  $\mathbb{Z}_{p-1} \times D_4$ , respectively. (Here  $D_4$  is the dihedral group of order 8.)

#### 15.2.2 Coherent subalgebras

We computed numbers of coherent subalgebras of  $\mathcal{M}$  which arise as algebraic mergings for  $p = 3, 5$  and  $7$ . We also attempted this for  $p = 11$ , but while isomorphism testing for certain coherent subalgebras required only 1-2 minutes, others had to be aborted, unsuccessfully, after



a two-week period. We therefore limit our table below to numbers of pairwise non-isomorphic coherent configurations obtained as algebraic mergings of  $\mathcal{M}$  for  $p = 3, 5$  and  $7$ .

$p$	CC	NCC	AS	Schur	NonSch	Intr
$p = 3$	22	12	10	8	2	0
$p = 5$	60	36	24	18	6	3
$p = 7$	120	80	40	28	12	4

We provide a legend for this table as follows: CC = number of coherent configurations, NCC = number of non-homogeneous coherent configurations, AS = number of association schemes, Schur = number of Schurian association schemes, NonSch = number of non-Schurian association schemes, Intr = number of non-Schurian association schemes with intransitive group of automorphisms.

We also arranged a number of computational experiments in order to enumerate all homogeneous mergings of  $\mathcal{M}$ , again appearing as algebraic mergings, for a few small values of  $p$ . The results will be reported elsewhere.

### 15.3 Proofs versus plausible reasonings

Plausible reasonings (exactly in the sense of G. Pólya [38, 39]) play a very significant role in AGT in general, and in this project concretely. The modern use of computers, of course, extends the possibilities of this method of mathematical thinking.

For decades, a striking model of research in AGT is to construct a new object with the aid of a computer, to understand the object's properties (ideally to create an aesthetically pleasing computer-free interpretation), and finally to extend the initial object to an infinite series (or even class) of new structures.

Fortunately, the authors are able to report exactly this level of success in this paper. Of course, *Pólya's method* (if fulfilled to its complete extent) ultimately requires rigorous proofs of all claims, thus substituting plausible insights by justified mathematical propositions.

The authors are definitely on track to do this. Up to Section 11, we reported rigorous results, confirmed at the level of accepted standards of mathematical rigour. Completion of the wider project and its exposition is simply a matter of time and space limitations. We hope that the reader is able to detect, and even appreciate, our intentions.

### 15.4 About the style of this paper

Our first and foremost task in this text is clear and traditional: to communicate reached new results. While the authors of the majority of research papers are typically satisfied by fulfilment of this goal, we were, from the very beginning, thinking about some additional objectives of our paper.

Taking into account that both arXiv and final versions are intended for online publication, space limitations are not so significant in such a case, although the reader's patience to read a longer text should definitely be taken under consideration. Striving to maintain a delicate balance between these two extrema, we briefly touch upon additional functions of communication of our text, formulated below as theses.

### 15.4.1 On the edge with philosophy of mathematics

**Education of a scientific researcher.** As previously mentioned, the younger coauthor started this project with a quite modest goal: to better explain known mathematical objects on 18 points. Fortunately for us, this explanation was quickly extended to an understanding of new larger objects, ultimately leading to formal theoretical generalisations. Both authors shared the enjoyment of this efficient maturation of ideas, and firmly believe that their process carries some unusual methodological features. Sharing of them might be helpful for other mathematical parties.

**Philosophy of discovery.** The computer was our closest ally at each stage of the conducted research project. All software tools at our disposal were available free of charge. Moreover, when it became absolutely necessary, we were able to communicate freely with colleagues who are high experts in the use of this software and benefit greatly from their advice. Although we could potentially conduct any number of experiments, usually 5-10 sufficed in order to guess a valid generalisation. At each stage it was a pleasure to experience a gradual deepening of our understanding of new occurrences.

**Role of proofs.** As a rule, we gained the main ingredients of our knowledge at a heuristic level. Typically, the elder coauthor was posing some quite natural questions in response to which the younger author was making the required calculations. A subsequent comparison of results would lead to the formulation of viable conjectures. With each successive stage of this methodological process, the younger colleague would come to acquire a deeper appreciation and comprehension of this kind of mathematical behaviour. Proofs were proceeded only after the corresponding picture became completely transparent to us, thus allowing the clearest possible formulation of claims, already evident to us.

**Advantages of a computer.** Once more, we were benefitting from the use of necessary free software. Typically, this was GAP with its share packages, COCO, COCO-IIR, and programs from the home page of Hanaki and Miyamoto.

We were also conducting experiments on how the resulting data should be presented and formatted, how tables could be organized for maximum benefit, making guesses of the structure of a group from its order, confirming assumptions about groups with the aid of generators, etc.

**Goals of mathematics.** This issue is on the edge with philosophy of science. One starts a very concrete project within the well established area of AGT, applies some traditional tools, and quickly becomes aware of the need to extend these tools in an innovative manner. The obtained results, which are very often quite far from would have been predicted from the outset, leads one to investigate links between the exploited area and other parts of mathematics.

What then should be regarded as the most significant result? Is it discovery, proof, beauty of the results, new research horizons? It is difficult to say. Probably a combination of all of these. This is how we naturally come to the forefront of philosophy of mathematics.

### 15.4.2 Extra vision from inside of philosophy of science

Not wishing to open this box of Pandora, we just mention here a handful of interesting references and excerpts, to which we add our own brief personal reflections.

Its purpose is to guide the reader to a better understanding of our style of exposition and our three main methodological activities: computer experiments, subsequent generalisations, and proofs that we often found tedious.

- The paper [43] played a very significant role in the long-standing discussion between mathematicians and philosophers, which is here very briefly touched. It contains thoughts of the late Fields medalist, with special attention to a “continuing desire for human understanding of a proof” and advocating to make them as clear and simple as possible.
- An interesting survey about the challenge of computer mathematics [2] culminated with a section entitled “Romantic versus cool mathematics”. Here *romantic* refers to proofs that are navigable by mind, while *cool* oppositely refers to proofs that are verified only by computer.
- The paper [9] is highly provocative from the outset, due to its Section 1 title: “To prove or not to prove – that is the question!”. The authors suggest to classify the evolution of a proof from ancient time to our millenium in eight stages, especially attributing to the eighth and final stage a much more significant role of empirical and experimental features.
- Paper [21] was written by a famous expert in didactics of mathematics, and is devoted to the role of formal proof in high school education. Its presented list of functions of proof, which includes verification, explanation, discovery, communication, etc., is fairly in concert with our own experience and philosophy, as they are discussed here.
- Our last item refers to an article [49] at the blog of a well known expert in computer algebra, and the creator and commercial promoter of **Mathematica**. It contains an interesting discussion of knowledge obtained via computer, and speculates about the future of the role of proof in pure mathematics.

## 15.5 Bosák graph

There is one more fairly well known graph on 18 vertices, which may be easily obtained inside of our master coherent configuration on 18 points. This graph belongs to the family of directed strongly regular graphs, briefly DSRGs, a natural generalisation of strongly regular (undirected) graphs to the case of mixed graphs. This concept was introduced by A. Duval in his seminal paper [12]. We will use the established notation  $(n, k, t, \lambda, \mu)$  for its parameter set.

The discussed graphs are regular graphs of valency  $k$ , and satisfy  $AJ = JA = kJ$  and  $A^2 = t \cdot I + \lambda \cdot A + \mu(J - I - A)$ . Clearly, always  $0 \leq t \leq k$ . If  $t = k$  then we are getting the usual strongly regular graphs, while the case  $t = 0$  corresponds to doubly regular tournaments. For  $0 < t < k$  the wording *genuine DSRG* was suggested.

The main ingredients of the theory of DSRGs were developed in [12] by Duval, who also discovered a few infinite classes of such graphs and posed problems of existence and full enumeration of all DSRGs, up to isomorphism, with a given parameter set. In particular, Duval mentioned the existence of a DSRG with the parameters  $(18, 4, 3, 0, 1)$ , and as well constructed an infinite family of such graphs with parameters  $(k^2 + k, k, 1, 0, 1)$ , where  $k \geq 2$ . It seems that Duval was unaware of the fact that his concrete graph on 18 vertices had already been discovered 10 years earlier by J. Bosák (see [5, 6, 7]), who was looking for so-called *mixed Moore graphs*. In

modern terms, these are DSRGs with  $\lambda = 0$  and  $\mu = 1$ , which appear as a natural generalisation of classical (undirected) Moore graphs. For this class of mixed graphs, Bosák developed a theory quite similar to the more general one established by Duval for all DSRGs later on.

It seems that for a long time Bosák’s results remained undetected by the majority of experts in AGT. Fortunately, the authors of [35] rediscovered Bosák’s publications and breathed new life into them. In particular, they proved uniqueness of DSRGs with parameters  $(18, 4, 3, 0, 1)$ , and replied to some open questions posed by Bosák. The results of Bosák were definitely not known to the authors of [15] and [25]. They, in fact, independently duplicated some of Bosák’s constructions in terms of 2-designs, paying special attention to the case  $(18, 4, 3, 0, 1)$ . Nowadays we refer to the graph with these parameters as the *Bosák graph*  $B_{18}$ . This graph is one of the main heroes in [25]; see Example 7.2 of that paper. The depiction of  $B_{18}$  provided there vividly shows that it has  $K_{3,3}$  as a quotient graph. Also it was shown in [25] that  $G = \text{Aut}(B_{18})$  has order 108, and an explicit set of generators for  $G$  was given. It was additionally observed that  $G$  is a central extension of  $\mathbb{Z}_3$  with the aid of the subgroup of index 2 in  $\text{Aut}(K_{3,3})$ , consisting of even permutations. (Incidentally, the original depiction of  $B_{18}$  given by Bosák in [7] carries very much the same flavour as the one given in [25], although we felt that the latter may be regarded as a bit more aesthetically pleasing.)

The graph  $B_{18}$  also attracted the attention of L. Jørgensen [24], who gave a description of  $B_{18}$  as a Cayley graph over a suitable (non-Abelian) group of order 18. We became aware of the references to Bosák from the preliminary version of [24], kindly sent to us by the author. This led the current authors to prepare the draft [18] in which  $B_{18}$  is considered in the framework of our master configuration  $\mathcal{M}$  on 18 points. In particular, we showed that the coherent closure of  $B_{18}$  is a certain rank 7 Schurian association scheme which arises as a merging of  $\mathcal{M}$ . The arc set of  $B_{18}$  is described as a union of relations of  $\mathcal{M}$ , specifically  $A_1$ ,  $C_2$ ,  $E_0$  and  $F_0$ . Removal of all directed edges from  $B_{18}$  leads to the Pappus graph. Also a new geometric image of  $B_{18}$  was created which reflects the embedding of the Bosák graph into the torus. One more interesting finding was a graphical representation of  $B_{18}$  with the aid of the voltage graph of order 2 in the group  $\mathbb{Z}_3 \times \mathbb{Z}_3$ .

Last but not least, we wish to mention that the draft [18] provided the author Š.Gy. a rare opportunity to enter into an additional research area of AGT. The recent joint publication [17] provides evidence, as well as hope, that his acquaintance with DSRGs is not a one-time affair.

## 15.6 Biaffine planes

The authors learned the term “biaffine plane” from the paper [47] of P. Wild, who used this terminology in a few of his earlier publications as well as in his Ph.D. thesis (University of London, 1980). Recently, however, we came to understand that the term was used quite earlier, namely in [37] by G. Pickert, one of the classic experts of modern finite geometries, who passed away in 2015 at the ripe old age of 97. The term is referred to M. Oehler (1975) and is discussed in the context of strongly regular graphs, namely the famous Shrikhande’s pseudo- $L_2$  association scheme.

A more extensive bibliographical search resulted in the discovery of a paper by A. Bennett [3] published in 1925. Quite in the style of that time the text [3] does not contain any references. Thus we urge experts in the field of history of finite geometries to determine the earliest origins of this terminology.

## 15.7 Spectra of our schemes

In Appendix 3, we provide a description of the spectrum of each of the four association schemes  $\mathcal{M}_1$ – $\mathcal{M}_4$  determined in this paper. As was discussed earlier, a justification of this result (as with similar results for rank 6 and rank 5 schemes) does not fit well the framework of this text. Nevertheless, we hope that by affording this spectral information to others, it may be of some definite help. We alert the reader that the last row in some tables still lacks a precise formulation.

## 15.8 A few extra references

We attract the reader's attention to a few recent publications which provide some interesting overlap with the topics touched here by us.

The paper [31] belongs to the area of coding theory. The introduced LDPC codes are based on an infinite family of bipartite graphs, known notationally as  $D(2, q)$ , which initially arose in the context of extremal graph theory (EGT). In [31] the spectrum of  $D(2, q)$  is determined, and it is proved that each such graph is Ramanujan. However, the reported spectrum is in error as it is not symmetric about 0, a theoretical requirement for bipartite graphs. Clearly, for  $q$  prime the graphs  $D(2, q)$  are living inside our scheme  $\mathcal{M}(q)$ . Thus it would be of special interest to compare the arguments in [31] with the data provided in our text.

The paper [8] deals with graphs related to cages. Here again the considered graphs are living inside our configuration  $\mathcal{M}_p$ , and it is shown that they are close to optimal in the framework of EGT.

Authors of the paper [10] describe the spectrum of the Wenger graphs  $W_m(q)$ . Some formulas for the multiplicities of the eigenvalues are provided. For certain cases, a comparison of these values to the ones that result from our own formulas might be of ample curiosity.

## 15.9 Presentations

Partial reports about the results expressed in this text were presented a few times:

- by M.K. at the workshop “84th workshop on general algebra”, Dresden, Germany, June 2012, see [http://tu-dresden.de/die\\_tu\\_dresden/fakultaeten/fakultaet\\_mathematik\\_und\\_naturwissenschaften/fachrichtung\\_mathematik/institute/algebra/aaa84/](http://tu-dresden.de/die_tu_dresden/fakultaeten/fakultaet_mathematik_und_naturwissenschaften/fachrichtung_mathematik/institute/algebra/aaa84/);
- by Š.Gy. at the conference “Computers in Scientific Discovery 6”, Portorož, Slovenia, 2012, see <http://conferences2.imfm.si/internalPage.py?pageId=15&confId=12>;
- by Š.Gy. at the workshop “50th Summer school on general algebra and ordered sets”, Nový Smokovec, Slovakia, Sept. 2012, see <https://sites.google.com/site/ssalgebra2012/>;
- by M.K. at the seminar of Queen Mary, University of London, England, May 2013, see <http://www.maths.qmul.ac.uk/seminars/>;
- by M.K. at the conference “Modern trends in AGT”, Villanova (PA), USA, June 2014, see <https://www1.villanova.edu/villanova/artsci/mathematics/newsevents/mtagt/slides-of-all-talks.html>.

Each of these presentations was very helpful for the authors, giving them the chance to better understand ways in which they could improve successive versions of the paper.

## 15.10 Research in progress

As was mentioned, we see a very promising intersection of our research methodology with activities in extremal graph theory. We are currently in a position to exploit this potential to its full extent. In particular, we aim to investigate how families of association schemes discovered by us (as well as other possible examples and classes of such structures) may be “repurposed” for beneficial application to other diverse branches of graph theory.

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## Appendix 1

In this appendix we would like to display the intersection numbers of the mentioned association schemes. For the sake of brevity let us denote  $\xi := \delta_{i+j,k} + \delta_{i-j,k} + \delta_{-i+j,k} + \delta_{-i-j,k}$ , and  $M_{ijk} := \max\{\delta_{i+j,k}, \delta_{i-j,k}\} + \max\{\delta_{-i+j,k}, \delta_{-i-j,k}\}$ .

The superscript in the upper left corner indicates which relation is fixed in the table. The subscripts  $i$  and  $j$  correspond to rows and columns, respectively, of the intersection matrices.

**Intersection numbers of  $\mathcal{M}_1$ :**

$c_{i,j}^0$	0	$S_j$	$T_j$	$U_j$
0	1	0	0	0
$S_i$	0	$\delta_{i,j'}$	0	0
$T_i$	0	0	$p \cdot \delta_{i,j'}$	0
$U_i$	0	0	0	$p \cdot \delta_{i,j'}$

$c_{i,j}^{sk}$	0	$S_j$	$T_j$	$U_j$
0	0	$\delta_{j,k}$	0	0
$S_i$	$\delta_{i,k}$	$\delta_{i+j,k}$	0	0
$T_i$	0	0	$p \cdot \delta_{i,j'}$	0
$U_i$	0	0	0	$p \cdot \delta_{i+j,k}$

$c_{i,j}^{tk}$	0	$S_j$	$T_j$	$U_j$
0	0	0	$\delta_{j,k}$	0
$S_i$	0	0	$\delta_{j,k}$	0
$T_i$	$\delta_{i,k}$	$\delta_{i,k}$	$p \cdot \delta_{i+j,k}$	0
$U_i$	0	0	0	1

$c_{i,j}^{uk}$	0	$S_j$	$T_j$	$U_j$
0	0	0	0	$\delta_{j,k}$
$S_i$	0	0	0	$\delta_{i+j,k}$
$T_i$	0	0	0	1
$U_i$	$\delta_{i,k}$	$\delta_{i+j,k}$	1	0

Intersection numbers of  $\mathcal{M}_2$ :

$c_{i,j}^0$	0	$S_j^*$	$T_j$	$U_0$	$U_j^*$
0	1	0	0	0	0
$S_i^*$	0	$2 \cdot \delta_{i,j}$	0	0	0
$T_i$	0	0	$p \cdot \delta_{i,j'}$	0	0
$U_0$	0	0	0	$p$	0
$U_i^*$	0	0	0	0	$2p \cdot \delta_{i,j}$

$c_{i,j}^{sk}$	0	$S_j^*$	$T_j$	$U_0$	$U_j^*$
0	0	$\delta_{j,k}$	0	0	0
$S_i^*$	$\delta_{i,k}$	$\xi$	0	0	0
$T_i$	0	0	$p \cdot \delta_{i,j'}$	0	0
$U_0$	0	0	0	$p \cdot \xi$	$p \cdot \xi$
$U_i^*$	0	0	0	$p \cdot \xi$	$p \cdot \xi$

$c_{i,j}^{tk}$	0	$S_j^*$	$T_j$	$U_0$	$U_j^*$
0	0	0	$\delta_{j,k}$	0	0
$S_i^*$	0	0	$2 \cdot \delta_{j,k}$	0	0
$T_i$	$\delta_{i,k}$	$2 \cdot \delta_{i,k}$	$p \cdot \delta_{i+j,k}$	0	0
$U_0$	0	0	0	1	2
$U_i^*$	0	0	0	2	4

$c_{i,j}^{uk}$	0	$S_j^*$	$T_j$	$U_0$	$U_j^*$
0	0	0	0	$\delta_{0,k}$	$\delta_{0,j}$
$S_i^*$	0	0	0	$M_{i0k}$	$M_{ijk}$
$T_i$	0	0	0	1	2
$U_0$	$\delta_{0,k}$	$M_{0jk}$	1	0	0
$U_i^*$	$\delta_{0,i}$	$M_{ijk}$	2	0	0

Intersection numbers of  $\mathcal{M}_3$ :

$c_{i,j}^0$	0	$S$	$T_j$	$U_0$	$U$
0	1	0	0	0	0
$S$	0	$p-1$	0	0	0
$T_i$	0	0	$p \cdot \delta_{i,j'}$	0	0
$U_0$	0	0	0	$p$	0
$U$	0	0	0	0	$p(p-1)$

$c_{i,j}^s$	0	$S$	$T_j$	$U_0$	$U$
0	0	1	0	0	0
$S$	1	$p-2$	0	0	0
$T_i$	0	0	$p \cdot \delta_{i,j'}$	0	0
$U_0$	0	0	0	0	$p$
$U$	0	0	0	$p$	$p(p-2)$

$c_{i,j}^{tk}$	0	$S$	$T_j$	$U_0$	$U$
0	0	0	$\delta_{j,k}$	0	0
$S$	0	0	$(p-1) \cdot \delta_{j,k}$	0	0
$T_i$	$\delta_{i,k}$	$(p-1) \cdot \delta_{i,k}$	$p \cdot \delta_{i+j,k}$	0	0
$U_0$	0	0	0	1	$p-1$
$U$	0	0	0	$p-1$	$(p-1)^2$

$c_{i,j}^{u0}$	0	$S$	$T_j$	$U_0$	$U$
0	0	0	0	1	0
$S$	0	0	0	0	$p-1$
$T_i$	0	0	0	1	$p-1$
$U_0$	1	0	1	0	0
$U$	0	$p-1$	$p-1$	0	0

$c_{i,j}^u$	0	$S$	$T_j$	$U_0$	$U$
0	0	0	0	0	1
$S$	0	0	0	1	$p-2$
$T_i$	0	0	0	1	$p-1$
$U_0$	0	1	1	0	0
$U$	1	$p-2$	$p-1$	0	0

Intersection numbers of  $\mathcal{M}_4$ :

$c_{i,j}^0$	0	$S$	$T_j^*$	$U_0$	$U$
0	1	0	0	0	0
$S$	0	$p-1$	0	0	0
$T_i^*$	0	0	$2p \cdot \delta_{i,j}$	0	0
$U_0$	0	0	0	$p$	0
$U$	0	0	0	0	$p(p-1)$

$c_{i,j}^s$	0	$S$	$T_j^*$	$U_0$	$U$
0	0	1	0	0	0
$S$	1	$p-2$	0	0	0
$T_i^*$	0	0	$2p \cdot \delta_{i,j}$	0	0
$U_0$	0	0	0	0	$p$
$U$	0	0	0	$p$	$p(p-2)$

$c_{i,j}^{tk}$	0	$S$	$T_j^*$	$U_0$	$U$
0	0	0	$\delta_{j,k}$	0	0
$S$	0	0	$(p-1) \cdot \delta_{j,k}$	0	0
$T_i^*$	$\delta_{i,k}$	$(p-1) \cdot \delta_{i,k}$	$p \cdot \xi$	0	0
$U_0$	0	0	0	1	$p-1$
$U$	0	0	0	$p-1$	$(p-1)^2$

$c_{i,j}^{u0}$	0	$S$	$T_j^*$	$U_0$	$U$
0	0	0	0	1	0
$S$	0	0	0	0	$p-1$
$T_i^*$	0	0	0	2	$2p-2$
$U_0$	1	0	2	0	0
$U$	0	$p-1$	$2p-2$	0	0

$c_{i,j}^u$	0	$S$	$T_j^*$	$U_0$	$U$
0	0	0	0	0	1
$S$	0	0	0	1	$p-2$
$T_i^*$	0	0	0	2	$2p-2$
$U_0$	0	1	2	0	0
$U$	1	$p-2$	$2p-2$	0	0

## Appendix 2

The table below reflects the manner in which we were able to recognize the existence of the non-Schurian association scheme  $\mathcal{N}_6$  based on analysis of computer data. The column headings indicate the value of  $p$ , the order of the scheme, the order of the automorphism group of the Schurian rank 8 scheme, the order of  $\text{Aut}(\mathcal{N}_6)$ , and whether or not  $\mathcal{N}_6$  is commutative, or symmetric.

$p$	$n$	rank 8	rank 6	commutative	symmetric
3	18	54	54	yes	no
5	50	1,000	1,000	yes	yes
7	98	6,174	6,174	yes	no
11	242	66,550	66,550	yes	no
13	338	158,184	158,184	yes	yes
17	578	628,864	628,864	yes	yes
19	722	1,111,158	1,111,158	yes	no

## Appendix 3: Spectrum of schemes $\mathcal{M}_1$ – $\mathcal{M}_4$

In all association schemes the spectrum of the identity matrix of order  $2p^2$  is omitted.

### Association scheme $\mathcal{M}_1$

There are  $p-1$  basic matrices with spectrum



eigenvalue	multiplicity
1	$2p$
roots of $1 + x + x^2 + \dots + x^{p-1}$	$2p$

$p - 1$  matrices with spectrum

eigenvalue	multiplicity
$p$	2
0	$2p(p - 1)$
roots of $\sum_{i=0}^{p-1} p^i \cdot x^{p-1-i}$	2

one matrix with spectrum

eigenvalue	multiplicity
$-p$	1
$p$	1
0	$2p - 2$
$-\sqrt{p}$	$p(p - 1)$
$\sqrt{p}$	$p(p - 1)$

$p - 1$  matrices with spectrum

eigenvalue	multiplicity
$-p$	1
$p$	1
0	$2p - 2$
roots of some polynomial of degree $2p - 2$	$p$

## Association scheme $\mathcal{M}_2$

There are  $(p - 1)/2$  matrices with spectrum

eigenvalue	multiplicity
$p - 1$	$2p$
roots of $x^{(p-1)/2} + x^{(p-3)/2} - \frac{p-3}{2}x^{(p-5)/2} - \frac{p-5}{2}x^{(p-7)/2} + \dots$	$4p$

$p - 1$  matrices with spectrum

eigenvalue	multiplicity
$p$	2
0	$2p(p - 1)$
roots of $\sum_{i=0}^{p-1} p^i \cdot x^{p-1-i}$	2

one matrix with spectrum

eigenvalue	multiplicity
$-p$	1
$p$	1
0	$2p - 2$
$-\sqrt{p}$	$p(p - 1)$
$\sqrt{p}$	$p(p - 1)$

$p - 1$  matrices with spectrum

eigenvalue	multiplicity
$-2p$	1
$2p$	1
0	$2p - 2$
roots of some polynomial of degree $p - 1$	$2p$

### Association scheme $\mathcal{M}_3$

There is one matrix with spectrum

eigenvalue	multiplicity
$p - 1$	$2p$
$-1$	$2p(p - 1)$

$p - 1$  matrices with spectrum

eigenvalue	multiplicity
$p$	2
0	$2p(p - 1)$
roots of $\sum_{i=0}^{p-1} p^i \cdot x^{p-1-i}$	2

one matrix with spectrum

eigenvalue	multiplicity
$-p$	1
$p$	1
0	$2p - 2$
$-\sqrt{p}$	$p(p - 1)$
$\sqrt{p}$	$p(p - 1)$

one matrix with spectrum

eigenvalue	multiplicity
$-p(p - 1)$	1
$-\sqrt{p}$	$p(p - 1)$
0	$2p - 2$
$\sqrt{p}$	$p(p - 1)$
$p(p - 1)$	1

## Association scheme $\mathcal{M}_4$

There is one matrix with spectrum

eigenvalue	multiplicity
$p - 1$	$2p$
$-1$	$2p(p - 1)$

$(p - 1)/2$  matrices with spectrum

eigenvalue	multiplicity
$2p$	$2$
$0$	$2p(p - 1)$
roots of some polynomial of degree $(p - 1)/2$	$4$

one matrix with spectrum

eigenvalue	multiplicity
$-p$	$1$
$p$	$1$
$0$	$2p - 2$
$-\sqrt{p}$	$p(p - 1)$
$\sqrt{p}$	$p(p - 1)$

one matrix with spectrum

eigenvalue	multiplicity
$-p(p - 1)$	$1$
$-\sqrt{p}$	$p(p - 1)$
$0$	$2p - 2$
$\sqrt{p}$	$p(p - 1)$
$p(p - 1)$	$1$

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